## Nonhomogeneous Linear Equations

**THEOREM 7** The general solution $y = y(x)$ to the nonhomogeneous differential equation (1) has the form

$$y = y_c + y_p,$$

where the **complementary solution** $y_c$ is the general solution to the associated homogeneous equation (2) and $y_p$ is any **particular solution** to the nonhomogeneous equation (1).

### The Method of Undetermined Coefficients

We first illustrate the method of undetermined coefficients for the equation

$$ay'' + by' + cy = G(x)$$

where $G(x)$ is a polynomial. It is reasonable to guess that there is a particular solution $y_p$ that is a polynomial of the same degree as $G$ because if $y$ is a polynomial, then $ay'' + by' + cy$ is also a polynomial. We therefore substitute $y_p(x)$ = a polynomial (of the same degree as $G$) into the differential equation and determine the coefficients.

**EXAMPLE 1** Solve the equation $y'' + y' - 2y = x^2$.

**SOLUTION** The auxiliary equation of $y'' + y' - 2y = 0$ is

$$r^2 + r - 2 = (r - 1)(r + 2) = 0$$

with roots $r = 1, -2$. So the solution of the complementary equation is

$$y_c = c_1e^x + c_2e^{-2x}$$

Since $G(x) = x^2$ is a polynomial of degree 2, we seek a particular solution of the form

$$y_p(x) = Ax^2 + Bx + C$$

Then $y_p' = 2Ax + B$ and $y_p'' = 2A$ so, substituting into the given differential equation, we have

$$(2A) + (2Ax + B) - 2(Ax^2 + Bx + C) = x^2$$

or

$$-2Ax^2 + (2A - 2B)x + (2A + B - 2C) = x^2$$

Polynomials are equal when their coefficients are equal. Thus

$$-2A = 1 \qquad 2A - 2B = 0 \qquad 2A + B - 2C = 0$$

The solution of this system of equations is

$$A = -\tfrac{1}{2} \qquad B = -\tfrac{1}{2} \qquad C = -\tfrac{3}{4}$$

A particular solution is therefore

$$y_p(x) = -\tfrac{1}{2}x^2 - \tfrac{1}{2}x - \tfrac{3}{4}$$

and, by Theorem 3, the general solution is

$$y = y_c + y_p = c_1e^x + c_2e^{-2x} - \tfrac{1}{2}x^2 - \tfrac{1}{2}x - \tfrac{3}{4}$$

If $G(x)$ (the right side of Equation 1) is of the form $Ce^{kx}$, where $C$ and $k$ are constants, then we take as a trial solution a function of the same form, $y_p(x) = Ae^{kx}$, because the derivatives of $e^{kx}$ are constant multiples of $e^{kx}$.

**EXAMPLE 2** Solve $y'' + 4y = e^{3x}$.

**SOLUTION** The auxiliary equation is $r^2 + 4 = 0$ with roots $\pm 2i$, so the solution of the complementary equation is

$$y_c(x) = c_1 \cos 2x + c_2 \sin 2x$$

For a particular solution we try $y_p(x) = Ae^{3x}$. Then $y_p' = 3Ae^{3x}$ and $y_p'' = 9Ae^{3x}$. Substituting into the differential equation, we have

$$9Ae^{3x} + 4(Ae^{3x}) = e^{3x}$$

so $13Ae^{3x} = e^{3x}$ and $A = \frac{1}{13}$. Thus, a particular solution is

$$y_p(x) = \tfrac{1}{13}e^{3x}$$

and the general solution is

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + \tfrac{1}{13}e^{3x}$$

If $G(x)$ is either $C \cos kx$ or $C \sin kx$, then, because of the rules for differentiating the sine and cosine functions, we take as a trial particular solution a function of the form

$$y_p(x) = A \cos kx + B \sin kx$$

**EXAMPLE 3** Solve $y'' + y' - 2y = \sin x$.

**SOLUTION** We try a particular solution

$$y_p(x) = A\cos x + B\sin x$$

Then $\qquad y_p' = -A\sin x + B\cos x \qquad y_p'' = -A\cos x - B\sin x$

so substitution in the differential equation gives

$$(-A\cos x - B\sin x) + (-A\sin x + B\cos x) - 2(A\cos x + B\sin x) = \sin x$$

or $$(-3A + B)\cos x + (-A - 3B)\sin x = \sin x$$

This is true if

$$-3A + B = 0 \qquad \text{and} \qquad -A - 3B = 1$$

The solution of this system is

$$A = -\tfrac{1}{10} \qquad B = -\tfrac{3}{10}$$

so a particular solution is

$$y_p(x) = -\tfrac{1}{10}\cos x - \tfrac{3}{10}\sin x$$

In Example 1 we determined that the solution of the complementary equation is $y_c = c_1e^x + c_2e^{-2x}$. Thus, the general solution of the given equation is

$$y(x) = c_1e^x + c_2e^{-2x} - \tfrac{1}{10}(\cos x + 3\sin x)$$

If $G(x)$ is a product of functions of the preceding types, then we take the trial solution to be a product of functions of the same type. For instance, in solving the differential equation

$$y'' + 2y' + 4y = x\cos 3x$$

we would try

$$y_p(x) = (Ax + B)\cos 3x + (Cx + D)\sin 3x$$

If $G(x)$ is a sum of functions of these types, we use the easily verified *principle of superposition,* which says that if $y_{p_1}$ and $y_{p_2}$ are solutions of

$$ay'' + by' + cy = G_1(x) \qquad ay'' + by' + cy = G_2(x)$$

respectively, then $y_{p_1} + y_{p_2}$ is a solution of

$$ay'' + by' + cy = G_1(x) + G_2(x)$$

**EXAMPLE 4** Solve $y'' - 4y = xe^x + \cos 2x$.

**SOLUTION** The auxiliary equation is $r^2 - 4 = 0$ with roots $\pm 2$, so the solution of the complementary equation is $y_c(x) = c_1 e^{2x} + c_2 e^{-2x}$. For the equation $y'' - 4y = xe^x$ we try

$$y_{p_1}(x) = (Ax + B)e^x$$

Then $y'_{p_1} = (Ax + A + B)e^x$, $y''_{p_1} = (Ax + 2A + B)e^x$, so substitution in the equation gives

$$(Ax + 2A + B)e^x - 4(Ax + B)e^x = xe^x$$

or
$$(-3Ax + 2A - 3B)e^x = xe^x$$

Thus, $-3A = 1$ and $2A - 3B = 0$, so $A = -\frac{1}{3}$, $B = -\frac{2}{9}$, and

$$y_{p_1}(x) = \left(-\tfrac{1}{3}x - \tfrac{2}{9}\right)e^x$$

For the equation $y'' - 4y = \cos 2x$, we try

$$y_{p_2}(x) = C\cos 2x + D\sin 2x$$

Substitution gives

$$-4C\cos 2x - 4D\sin 2x - 4(C\cos 2x + D\sin 2x) = \cos 2x$$

or
$$-8C\cos 2x - 8D\sin 2x = \cos 2x$$

Therefore, $-8C = 1$, $-8D = 0$, and

$$y_{p_2}(x) = -\tfrac{1}{8}\cos 2x$$

By the superposition principle, the general solution is

$$y = y_c + y_{p_1} + y_{p_2} = c_1 e^{2x} + c_2 e^{-2x} - \left(\tfrac{1}{3}x + \tfrac{2}{9}\right)e^x - \tfrac{1}{8}\cos 2x$$

Finally we note that the recommended trial solution $y_p$ sometimes turns out to be a solution of the complementary equation and therefore can't be a solution of the nonhomogeneous equation. In such cases we multiply the recommended trial solution by $x$ (or by $x^2$ if necessary) so that no term in $y_p(x)$ is a solution of the complementary equation.

**EXAMPLE 5** Solve $y'' + y = \sin x$.

**SOLUTION** The auxiliary equation is $r^2 + 1 = 0$ with roots $\pm i$, so the solution of the complementary equation is

$$y_c(x) = c_1 \cos x + c_2 \sin x$$

Ordinarily, we would use the trial solution

$$y_p(x) = A \cos x + B \sin x$$

but we observe that it is a solution of the complementary equation, so instead we try

$$y_p(x) = Ax \cos x + Bx \sin x$$

Then
$$y_p'(x) = A \cos x - Ax \sin x + B \sin x + Bx \cos x$$
$$y_p''(x) = -2A \sin x - Ax \cos x + 2B \cos x - Bx \sin x$$

Substitution in the differential equation gives

$$y_p'' + y_p = -2A \sin x + 2B \cos x = \sin x$$

so $A = -\frac{1}{2}$, $B = 0$, and

$$y_p(x) = -\tfrac{1}{2}x \cos x$$

The general solution is

$$y(x) = c_1 \cos x + c_2 \sin x - \tfrac{1}{2}x \cos x$$

We summarize the method of undetermined coefficients as follows:

1. If $G(x) = e^{kx}P(x)$, where $P$ is a polynomial of degree $n$, then try $y_p(x) = e^{kx}Q(x)$, where $Q(x)$ is an $n$th-degree polynomial (whose coefficients are determined by substituting in the differential equation.)
2. If $G(x) = e^{kx}P(x) \cos mx$ or $G(x) = e^{kx}P(x) \sin mx$, where $P$ is an $n$th-degree polynomial, then try

$$y_p(x) = e^{kx}Q(x) \cos mx + e^{kx}R(x) \sin mx$$

where $Q$ and $R$ are $n$th-degree polynomials.

*Modification:* If any term of $y_p$ is a solution of the complementary equation, multiply $y_p$ by $x$ (or by $x^2$ if necessary).

**EXAMPLE 6** Determine the form of the trial solution for the differential equation $y'' - 4y' + 13y = e^{2x} \cos 3x$.

**SOLUTION** Here $G(x)$ has the form of part 2 of the summary, where $k = 2$, $m = 3$, and $P(x) = 1$. So, at first glance, the form of the trial solution would be

$$y_p(x) = e^{2x}(A \cos 3x + B \sin 3x)$$

But the auxiliary equation is $r^2 - 4r + 13 = 0$, with roots $r = 2 \pm 3i$, so the solution of the complementary equation is

$$y_c(x) = e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$$

This means that we have to multiply the suggested trial solution by $x$. So, instead, we use

$$y_p(x) = xe^{2x}(A \cos 3x + B \sin 3x)$$

| $r(x)$ | **Initial guess for $y_p(x)$** |
|---|---|
| $k$ (a constant) | $A$ (a constant) |
| $ax + b$ | $Ax + B$ (*Note*: The guess must include both terms even if $b =$ |
| $ax^2 + bx + c$ | $Ax^2 + Bx + C$ (*Note*: The guess must include all three terms if $b$ or $c$ are zero.) |
| Higher-order polynomials | Polynomial of the same order as $r(x)$ |
| $ae^{\lambda x}$ | $Ae^{\lambda x}$ |
| $a \cos \beta x + b \sin \beta x$ | $A \cos \beta x + B \sin \beta x$ (*Note*: The guess must include both term even if either $a = 0$ or $b = 0$.) |
| $ae^{\alpha x} \cos \beta x + be^{\alpha x} \sin \beta x$ | $Ae^{\alpha x} \cos \beta x + Be^{\alpha x} \sin \beta x$ |
| $(ax^2 + bx + c)\, e^{\lambda x}$ | $(Ax^2 + Bx + C)\, e^{\lambda x}$ |
| $(a_2x^2 + a_1x + a_0) \cos \beta x + (b_2x^2 + b_1x + b_0) \sin \beta x$ | $(A_2x^2 + A_1x + A_0) \cos \beta x + (B_2x^2 + B_1x + B_0) \sin \beta x$ |
| $(a_2x^2 + a_1x + a_0)\, e^{\alpha x} \cos \beta x + (b_2x^2 + b_1x + b_0)\, e^{\alpha x} \sin \beta x$ | $(A_2x^2 + A_1x + A_0)\, e^{\alpha x} \cos \beta x + (B_2x^2 + B_1x + B_0)\, e^{\alpha x} \sin \beta x$ |

## Variation of Parameters

Sometimes, $r(x)$ is not a combination of polynomials, exponentials, or sines and cosines. When this is tl the method of undetermined coefficients does not work, and we have to use another approach to find a particular solution to the differential equation. We use an approach called the **method of variation of parameters.**

To simplify our calculations a little, we are going to divide the differential equation through by $a$, so we ha leading coefficient of 1. Then the differential equation has the form

$$y'' + py' + qy = r(x),$$

where $p$ and $q$ are constants.

If the general solution to the complementary equation is given by $c_1 y_1(x) + c_2 y_2(x)$, we are going to lc particular solution of the form $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$. In this case, we use the two linearly independent solutions to the complementary equation to form our particular solution. However, we are as the coefficients are functions of *x*, rather than constants. We want to find functions $u(x)$ and $v(x)$ such t $y_p(x)$ satisfies the differential equation. We have

$$\begin{aligned} y_p &= uy_1 + vy_2 \\ y_p' &= u'y_1 + uy_1' + v'y_2 + vy_2' \\ y_p'' &= (u'y_1 + v'y_2)' + u'y_1' + uy_1'' + v'y_2' + vy_2''. \end{aligned}$$

Substituting into the differential equation, we obtain

$$\begin{aligned} y_p'' + py_p' + qy_p &= \left[(u'y_1 + v'y_2)' + u'y_1' + uy_1'' + v'y_2' + vy_2''\right] \\ &\quad + p\left[u'y_1 + uy_1' + v'y_2 + vy_2'\right] + q\left[uy_1 + vy_2\right] \\ &= u\left[y_1'' + py_1' + qy_1\right] + v\left[y_2'' + py_2' + qy_2\right] \\ &\quad + (u'y_1 + v'y_2)' + p\left(u'y_1 + v'y_2\right) + (u'y_1' + v'y_2'). \end{aligned}$$

Note that $y_1$ and $y_2$ are solutions to the complementary equation, so the first two terms are zero. Thus, v

$$(u'y_1 + v'y_2)' + p\left(u'y_1 + v'y_2\right) + (u'y_1' + v'y_2') = r(x).$$

If we simplify this equation by imposing the additional condition $u'y_1 + v'y_2 = 0$, the first two terms are and this reduces to $u'y_1' + v'y_2' = r(x)$. So, with this additional condition, we have a system of two ec in two unknowns:

$$\begin{aligned} u'y_1 + v'y_2 &= 0 \\ u'y_1' + v'y_2' &= r(x). \end{aligned}$$

Solving this system gives us $u'$ and $v'$, which we can integrate to find *u* and *v*.

Then, $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$ is a particular solution to the differential equation. Solving this s of equations is sometimes challenging, so let's take this opportunity to review Cramer's rule, which allows solve the system of equations using determinants.

**RULE: CRAMER'S RULE**

The system of equations

$$\begin{aligned} a_1 z_1 + b_1 z_2 &= r_1 \\ a_2 z_1 + b_2 z_2 &= r_2 \end{aligned}$$

has a unique solution if and only if the determinant of the coefficients is not zero. In this case, the solution is given by

$$z_1 = \frac{\begin{vmatrix} r_1 & b_1 \\ r_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \quad \text{and} \quad z_2 = \frac{\begin{vmatrix} a_1 & r_1 \\ a_2 & r_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

Use Cramer's rule to solve the following system of equations.

$$\begin{aligned} x^2 z_1 + 2x z_2 &= 0 \\ z_1 - 3x^2 z_2 &= 2x \end{aligned}$$

[Show/Hide Solution]

**Solution**

We have

$$\begin{aligned} a_1(x) &= x^2 \\ a_2(x) &= 1 \\ b_1(x) &= 2x \\ b_2(x) &= -3x^2 \\ r_1(x) &= 0 \\ r_2(x) &= 2x. \end{aligned}$$

Then,

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} x^2 & 2x \\ 1 & -3x^2 \end{vmatrix} = -3x^4 - 2x$$

$$\begin{vmatrix} r_1 & b_1 \\ r_2 & b_2 \end{vmatrix} = \begin{vmatrix} 0 & 2x \\ 2x & -3x^2 \end{vmatrix} = 0 - 4x^2 = -4x^2.$$

Thus,

$$z_1 = \frac{\begin{vmatrix} r_1 & b_1 \\ r_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{-4x^2}{-3x^4 - 2x} = \frac{4x}{3x^3 + 2}.$$

In addition,

$$\begin{vmatrix} a_1 & r_1 \\ a_2 & r_2 \end{vmatrix} = \begin{vmatrix} x^2 & 0 \\ 1 & 2x \end{vmatrix} = 2x^3 - 0 = 2x^3.$$

Thus,

$$z_2 = \frac{\begin{vmatrix} a_1 & r_1 \\ a_2 & r_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{2x^3}{-3x^4 - 2x} = \frac{-2x^2}{3x^3 + 2}.$$

## Home Work

Use Cramer's rule to solve the following system of equations.

$$\begin{aligned} 2xz_1 - 3z_2 &= 0 \\ x^2 z_1 + 4xz_2 &= x + 1 \end{aligned}$$

### PROBLEM-SOLVING STRATEGY

**Problem-Solving Strategy: Method of Variation of Parameters**

1. Solve the complementary equation and write down the general solution

$$c_1 y_1(x) + c_2 y_2(x).$$

2. Use Cramer's rule or another suitable technique to find functions $u'(x)$ and $v'(x)$ satisfying

$$\begin{aligned} u'y_1 + v'y_2 &= 0 \\ u'y_1{}' + v'y_2{}' &= r(x). \end{aligned}$$

3. Integrate $u'$ and $v'$ to find $u(x)$ and $v(x)$. Then, $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$ is a particular solution to the equation.
4. Add the general solution to the complementary equation and the particular solution found in st 3 to obtain the general solution to the nonhomogeneous equation.

**EXAMPLE 7.16**

### Using the Method of Variation of Parameters

Find the general solution to the following differential equations.

a. $y'' - 2y' + y = \frac{e^t}{t^2}$
b. $y'' + y = 3\sin^2 x$

[Show/Hide Solution]

### Solution

a. The complementary equation is $y'' - 2y' + y = 0$ with associated general solution $c_1 e^t + c_2 te^t$. Therefore, $y_1(t) = e^t$ and $y_2(t) = te^t$. Calculating the derivatives, we get $y_1'(t) = e^t$ and $y_2'(t) = e^t + te^t$ (step 1). Then, we want to find functions $u'(t)$ and $v'(t)$ so that

$$\begin{aligned} u'e^t + v'te^t &= 0 \\ u'e^t + v'\left(e^t + te^t\right) &= \frac{e^t}{t^2}. \end{aligned}$$

Applying Cramer's rule, we have

$$u' = \frac{\begin{vmatrix} 0 & te^t \\ \frac{e^t}{t^2} & e^t + te^t \end{vmatrix}}{\begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix}} = \frac{0 - te^t\left(\frac{e^t}{t^2}\right)}{e^t\left(e^t + te^t\right) - e^t te^t} = \frac{-\frac{e^{2t}}{t}}{e^{2t}} = -\frac{1}{t}$$

$$v' = \frac{\begin{vmatrix} e^t & 0 \\ e^t & \frac{e^t}{t^2} \end{vmatrix}}{\begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix}} = \frac{e^t\left(\frac{e^t}{t^2}\right)}{e^{2t}} = \frac{1}{t^2} \text{ (step 2)}.$$

Integrating, we get

$$\begin{aligned} u &= -\int \frac{1}{t}dt = -\ln|t| \\ v &= \int \frac{1}{t^2}dt = -\frac{1}{t} \text{ (step 3)}. \end{aligned}$$

Then we have

$$\begin{aligned} y_p &= -e^t \ln|t| - \frac{1}{t}te^t \\ &= -e^t \ln|t| - e^t \text{ (step 4)}. \end{aligned}$$

The $e^t$ term is a solution to the complementary equation, so we don't need to carry that term into our general solution explicitly. The general solution is

$$y(t) = c_1 e^t + c_2 te^t - e^t \ln|t| \text{ (step 5)}.$$

b. The complementary equation is $y'' + y = 0$ with associated general solution $c_1 \cos x + c_2 \sin x$. So, $y_1(x) = \cos x$ and $y_2(x) = \sin x$ (step 1). Then, we want to find functions $u'(x)$ and $v'(x)$ such that

$$\begin{aligned} u' \cos x + v' \sin x &= 0 \\ -u' \sin x + v' \cos x &= 3 \sin^2 x. \end{aligned}$$

Applying Cramer's rule, we have

$$u' = \frac{\begin{vmatrix} 0 & \sin x \\ 3\sin^2 x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{0 - 3\sin^3 x}{\cos^2 x + \sin^2 x} = -3\sin^3 x$$

and

$$v' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & 3\sin^2 x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{3\sin^2 x \cos x}{1} = 3\sin^2 x \cos x \text{ (step 2)}.$$

Integrating first to find $u$, we get

$$u = \int -3\sin^3 x\, dx = -3\left[-\tfrac{1}{3}\sin^2 x \cos x + \tfrac{2}{3}\int \sin x\, dx\right] = \sin^2 x \cos x + 2\cos x.$$

Now, we integrate to find $v$. Using substitution (with $w = \sin x$), we get

$$v = \int 3\sin^2 x \cos x\, dx = \int 3w^2\, dw = w^3 = \sin^3 x.$$

Then,

$$\begin{aligned} y_p &= (\sin^2 x \cos x + 2\cos x)\cos x + (\sin^3 x)\sin x \\ &= \sin^2 x \cos^2 x + 2\cos^2 x + \sin^4 x \\ &= 2\cos^2 x + \sin^2 x(\cos^2 x + \sin^2 x) \qquad \text{(step 4)}. \\ &= 2\cos^2 x + \sin^2 x \\ &= \cos^2 x + 1 \end{aligned}$$

The general solution is

$$y(x) = c_1 \cos x + c_2 \sin x + 1 + \cos^2 x \text{ (step 5)}.$$

**CHECKPOINT 7.14**

Find the general solution to the following differential equations.

a. $y'' + y = \sec x$
b. $x'' - 2x' + x = \frac{e^t}{t}$

## Section 7.2 Exercises

Solve the following equations using the method of undetermined coefficients.

54. $2y'' - 5y' - 12y = 6$

55. $3y'' + y' - 4y = 8$

56. $y'' - 6y' + 5y = e^{-x}$

57. $y'' + 16y = e^{-2x}$

58. $y'' - 4y = x^2 + 1$

59. $y'' - 4y' + 4y = 8x^2 + 4x$

# 6.5 Cauchy–Euler Equation

**Cauchy–Euler equation**

One of the simplest linear variable coefficient differential equations is the homogeneous second order **Cauchy–Euler** equation, whose standard form is

$$x^2\frac{d^2y}{dx^2} + a_1x\frac{dy}{dx} + a_2y = 0. \qquad (55)$$

The solution of this homogeneous equation can be reduced to a simple algebraic problem by seeking a solution of the form

$$y(x) = Ax^m, \qquad (56)$$

where $A$ is an arbitrary constant, and the permissible values of $m$ are to be determined.

Differentiating $y(x)$ to obtain

$$\frac{dy}{dx} = mAx^{m-1} \quad \text{and} \quad \frac{d^2y}{dx^2} = m(m-1)Ax^{m-2} \qquad (57)$$

and substituting these expressions into the Cauchy–Euler equation gives the following quadratic equation for $m$:

$$m(m-1) + a_1m + a_2 = 0. \qquad (58)$$

When this equation has two distinct real roots $m = \alpha$ and $m = \beta$, the general solution of (55) is

$$y(x) = C_1 x^{\alpha} + C_2 x^{\beta}, \tag{59}$$

but if the two roots are real and equal with $m = \mu$, the general solution of (55) is

$$y(x) = C_1 x^{\mu} + C_2 x^{\mu} \ln|x|, \tag{60}$$

where $C_1$ and $C_2$ are arbitrary real constants.

If the equation for $m$ has the complex conjugate roots $m = \alpha \pm i\beta$, substitution confirms that the general solution of (55) is

$$y(x) = C_1 x^{\alpha} \cos(\beta \ln|x|) + C_2 x^{\alpha} \sin(\beta \ln|x|). \tag{61}$$

The second solution $x^{\mu} \ln|x|$ in (60) can be obtained from the method of Section 6.7 by using the known solution $y_1(x) = x^{\mu}$ to find a second linearly independent solution $y_2(x)$. The form of solution (61) follows from writing the general solution as $y(x) = A\exp(\alpha + i\beta) + B\exp(\alpha - i\beta)$, with $A$ an arbitrary complex constant and $B$ its complex conjugate so that $y(x)$ is real.

**EXAMPLE 6.16** Find the general solution of

$$x^2 \frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 2y = 0 \quad \text{for } x \neq 0.$$

***Solution*** The equation for $m$ is

$$m(m-1) + 3m + 2 = 0,$$

with the roots $m = -1 \pm i$. The general solution is thus

$$y(x) = C_1 x^{-1} \cos(\ln|x|) + C_2 x^{-1} \sin(\ln|x|).$$

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## Example

Solve differential equation $x^2y'' - 3xy' + 4y = 0$ with initial values $y(1) = 1$ and $y'(1) = 2$.

We assume that the solution is in the form $y(x) = x^m$. Then

$$\begin{aligned} y(x) &= x^m, \\ y'(x) &= mx^{m-1}, \\ y''(x) &= m(m-1)x^{m-2}. \end{aligned}$$

Let us use the solution within given differential equation:

$$\begin{aligned} x^2 \cdot m(m-1)x^{m-2} - 3x \cdot mx^{m-1} + 4 \cdot x^m &= 0 \\ x^m((m^2 - m) - 3m + 4) &= 0 \\ x^m(m^2 - 4m + 4) &= 0 \\ m &= \{2, 2\} \end{aligned}$$

The solution of differential equation is

$$\underline{y_c(x) = c_1 x^2 + c_2 x^2 \log x.}$$

In order to find values of $c_1$ and $c_2$ according to initial values, we need also $y_c'$:

$$\begin{aligned} y_c'(x) &= 2c_1 x + 2c_2 x \log x + c_2 x^2 \frac{1}{x} = \\ &= 2c_1 x + 2c_2 x \log x + c_2 x. \end{aligned}$$

For $y(1) = 1$ and $y'(1) = 2$ we get from (1) and (2) $c_1 = 1, c_2 = 0$. The solution is then

$$\underline{\underline{y(x) = x^2.}}$$

## Example

Solve differential equation $x^2 y'' - 2xy' + 2y = x^3, \quad x > 0$

Because the differential equation is in the form of Cauchy-Euler DE we assume that the solution is in the form of $y(x) = x^m$. Then

$$\begin{aligned} y(x) &= x^m, \\ y'(x) &= mx^{m-1}, \\ y''(x) &= m(m-1)x^{m-2}. \end{aligned}$$

Let us use the solution within associated homogeneous differential equation to obtain complementary function $y_c(x)$:

$$\begin{aligned} x^2 \cdot m(m-1)x^{m-2} - 2x \cdot mx^{m-1} + 2 \cdot x^m &= 0 \\ x^m(m(m-1) - 2m + 2) &= 0 \\ x^m(m^2 - 3m + 2) &= 0 \\ x^m(m-2)(m-1) &= 0 \\ m &= \{1, 2\} \end{aligned}$$

The complementary function $y_c(x)$ (i.e. the solution of associated homogeneous differential equation) is

$$\underline{y_c(x) = c_1 x + c_2 x^2.}$$

## Example

Solve differential equation $x^2y'' + 4xy' - 4y = 3$

First, let us solve associated homogeneous DE $x^2y'' + 4xy' - 4y = 0$ for $y_c$, then particular solution $y_p$.

We expect the solution in a form of $y = x^m$, then $y' = mx^{m-1},\ y'' = m(m-1)x^{m-2}$.

$$x^2\ m(m-1)x^{m-2} + 4x\ mx^{m-1} - 4\ x^m = 0$$
$$x^m[m(m-1) + 4m - 4] = 0$$
$$x^m(m^2 + 3m - 4) = 0 \implies m = \{-4, 1\}$$

The complementary function is $y_c = c_1x + c_2x^{-4}$ and particular solution $y_p = A$ has to be determined and it is pretty obvious that $A = -3/4$ (test yourself that $y_p = -3/4$ satisfies given DE).

$$y = c_1x + c_2x^{-4} - \frac{3}{4}$$

Find the general solution: $x^2y'' + 5xy' + 12y = 0$

## Solution

As with the constant coefficient equations, we begin by writing down the characteristic equation. Doing a simple computation,

$$\begin{aligned} 0 &= r(r-1) + 5r + 12 \\ &= r^2 + 4r + 12 \\ &= (r+2)^2 + 8, \\ -8 &= (r+2)^2, \end{aligned}$$

one determines the roots are $r = -2 \pm 2\sqrt{2}i$. Therefore, the general solution is

$$y(x) = [c_1 \cos(2\sqrt{2}\ln|x|) + c_2 \sin(2\sqrt{2}\ln|x|)]\ x^{-2}$$

Find the general solution: $x^2y'' + 5xy' + 12y = 0$

## Solution

As with the constant coefficient equations, we begin by writing down the characteristic equation. Doing a simple computation,

$$\begin{aligned} 0 &= r(r-1) + 5r + 12 \\ &= r^2 + 4r + 12 \\ &= (r+2)^2 + 8, \\ -8 &= (r+2)^2, \end{aligned}$$

one determines the roots are $r = -2 \pm 2\sqrt{2}i$. Therefore, the general solution is

$$y(x) = \left[c_1 \cos(2\sqrt{2}\ln|x|) + c_2 \sin(2\sqrt{2}\ln|x|)\right] x^{-2}$$

Deriving the solution for Case 2 for the Cauchy-Euler equations works in the same way as the second for constant coefficient equations, but it is a bit messier. First note that for the real root, $r = r_1$, the characteristic equation has to factor as $(r - r_1)^2 = 0$. Expanding, we have

$$r^2 - 2r_1 r + r_1^2 = 0$$

$$0 = y(1) = c_1$$

Thus, we have so far that $y(t) = c_2 \ln|t| t^{-1}$.

Now, using the second condition and

$$y'(t) = c_2(1 - \ln|t|)t^{-2}$$

we have

$$1 = y(1) = c_2$$

Therefore, the solution of the initial value problem is $y(t) = \ln|t| t^{-1}$.

We now turn to the case of complex conjugate roots, $r = \alpha \pm i\beta$. When dealing with the Cauchy-Euler equations, we have solutions of the form $y(x) = x^{\alpha + i\beta}$. The key to obtaining real solutions is to first rewrite $x^y$ :

$$x^y = e^{\ln x^y} = e^{y \ln x}$$

Thus, a power can be written as an exponential and the solution can be written as

$$y(x) = x^{\alpha + i\beta} = x^{\alpha} e^{i\beta \ln x}, \quad x > 0$$

For complex conjugate roots, $r = \alpha \pm i\beta$, the general solution takes the form $y(x) = x^{\alpha}\left(c_1 \cos(\beta \ln|x|) + c_2 \sin(\beta \ln|x|)\right)$

Recalling that

$$e^{i\beta \ln x} = \cos(\beta \ln|x|) + i \sin(\beta \ln|x|),$$

we can now find two real, linearly independent solutions, $x^{\alpha} \cos(\beta \ln|x|)$ and $x^{\alpha} \sin(\beta \ln|x|)$ following the same steps as earlier for the constant coefficient case. This gives the general solution as

$$y(x) = x^{\alpha}\left(c_1 \cos(\beta \ln|x|) + c_2 \sin(\beta \ln|x|)\right)$$

✔ Example 2.5.3

Solve: $x^2y'' - xy' + 5y = 0$.

### Solution

The characteristic equation takes the form

$$r(r-1) - r + 5 = 0$$

or

$$r^2 - 2r + 5 = 0$$

The roots of this equation are complex, $r_{1,2} = 1 \pm 2i$. Therefore, the general solution is $y(x) = x\left(c_1 \cos(2\ln|x|) + c_2 \sin(2\ln|x|)\right)$.

e three cases are summarized below.

Classification of Roots of the Characteristic Equation for Cauchy-Euler Differential Equations

1. Real, distinct roots $r_1, r_2$. In this case the solutions corresponding to each root are linearly independent. Therefore, the general solution is simply

$$y(x) = c_1 x^{r_1} + c_2 x^{r_2}.$$

2. Real, equal roots $r_1 = r_2 = r$. In this case the solutions corresponding to each root are linearly dependent. To find a second linearly independent solution, one uses the Method of Reduction of Order. This gives the second solution as $x^r \ln|x|$. Therefore, the general solution is found as

$$y(x) = (c_1 + c_2 \ln|x|)\, x^r.$$

3. Complex conjugate roots $r_1, r_2 = \alpha \pm i\beta$. In this case the solutions corresponding to each root are linearly independent. These complex exponentials can be rewritten in terms of trigonometric functions. Namely, one has that $x^\alpha \cos(\beta \ln|x|)$ and $x^\alpha \sin(\beta \ln|x|)$ are two linearly independent solutions. Therefore, the general solution becomes

$$y(x) = x^\alpha \left(c_1 \cos(\beta \ln|x|) + c_2 \sin(\beta \ln|x|)\right).$$

## Nonhomogeneous Cauchy-Euler Equations

We can also solve some nonhomogeneous Cauchy-Euler equations using the Method of Undetermined Coefficients or the Method of Variation of Parameters. We will demonstrate this with a couple of examples.

✔ Example 2.5.4

Find the solution of $x^2y'' - xy' - 3y = 2x^2$.

### Solution

First we find the solution of the homogeneous equation. The characteristic equation is $r^2 - 2r - 3 = 0$. So, the roots are $r = -1, 3$ and the solution is $y_h(x) = c_1 x^{-1} + c_2 x^3$.

We next need a particular solution. Let's guess $y_p(x) = Ax^2$. Inserting the guess into the nonhomogeneous differential equation, we have

$$\begin{aligned} 2x^2 = x^2y'' - xy' - 3y = 2x^2 \\ = 2Ax^2 - 2Ax^2 - 3Ax^2 \\ = -3Ax^2 \end{aligned}$$

So, $A = -2/3$. Therefore, the general solution of the problem is

$$y(x) = c_1 x^{-1} + c_2 x^3 - \frac{2}{3} x^2$$

Find the solution of $x^2 y'' - xy' - 3y = 2x^3$.

### Solution

In this case the nonhomogeneous term is a solution of the homogeneous problem, which we solved in the last example. So, we will need a modification of the method. We have a problem of the form

$$ax^2 y'' + bxy' + cy = dx^r$$

where $r$ is a solution of $ar(r-1) + br + c = 0$. Let's guess a solution of the form $y = Ax^r \ln x$. Then one finds that the differential equation reduces to $Ax^r(2ar - a + b) = dx^r$. [You should verify this for yourself.]

With this in mind, we can now solve the problem at hand. Let $y_p = Ax^3 \ln x$. Inserting into the equation, we obtain $4Ax^3 = 2x^3$, or $A = 1/2$. The general solution of the problem can now be written as

$$y(x) = c_1 x^{-1} + c_2 x^3 + \frac{1}{2} x^3 \ln x$$

Find the solution of $x^2 y'' - xy' - 3y = 2x^3$ using Variation of Parameters.

### Solution

As noted in the previous examples, the solution of the homogeneous problem has two linearly independent solutions, $y_1(x) = x^{-1}$ and $y_2(x) = x^3$. Assuming a particular solution of the form $y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$, we need to

solve the system 2.4.25:

$$c_1'(x)x^{-1} + c_2'(x)x^3 = 0$$

$$-c_1'(x)x^{-2} + 3c_2'(x)x^2 = \frac{2x^3}{x^2} = 2x.$$

From the first equation of the system we have $c_1'(x) = -x^4 c_2'(x)$. Substituting this into the second equation gives $c_2'(x) = \frac{1}{2x}$. So, $c_2(x) = \frac{1}{2} \ln |x|$ and, therefore, $c_1(x) = \frac{1}{8} x^4$. The particular solution is

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x) = \frac{1}{8} x^3 + \frac{1}{2} x^3 \ln |x|$$

Adding this to the homogeneous solution, we obtain the same solution as in the last example using the Method of Undetermined Coefficients. However, since $\frac{1}{8} x^3$ is a solution of the homogeneous problem, it can be absorbed into the first terms, leaving

$$y(x) = c_1 x^{-1} + c_2 x^3 + \frac{1}{2} x^3 \ln x$$