Second and Higher Order Linear Differential Equations and Systems

Introduction

Linear second order differential equations with constant coefficients are the simplest of the higher order differential equations, and they have many applications. They are of the general form $y'' + Ay' + By = F(x)$ with A and B constants and $F(x)$, called the nonhomogeneous term, a known function of x. The equation is called nonhomogeneous when $F(x)$ is not identically zero; otherwise, it is called homogeneous. All general solutions are shown to be the sum of two quite different parts, one being a solution of the homogeneous equation called the complementary function that contains the expected two arbitrary constants of integration, and the other a special solution called a particular integral that depends only on $F(x)$ and contains no arbitrary constants.

6.1 Homogeneous Linear Constant Coefficient Second Order Equations

linear constant coefficient second order equation

The simplest general higher order homogeneous differential equation that occurs in applications is the **linear constant coefficient second order equation**

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0. \quad (1)$$

general solution

arbitrary constants, the **general solution** of (1) from which all particular solutions can be obtained can be written

$$y(x) = c_1 y_1(x) + c_2 y_2(x). \quad (5)$$

EXAMPLE 6.2

Direct substitution of the functions $y_1(x) = \sin 2x$ and $y_2(x) = \cos 2x$ into the second order differential equation

$$y'' + 4y = 0$$

confirms that they are solutions. The functions are linearly independent for all x because they are not proportional, so

$$y(x) = c_1 \cos 2x + c_2 \sin 2x$$

is the general solution of the differential equation. ■

We will now find the general solution of (1), and when doing so use will be made of the fact that if $y(x) = ce^{\lambda x}$, with c and λ constants, then

$$\frac{dy}{dx} = \frac{d[ce^{\lambda x}]}{dx} = c\lambda e^{\lambda x} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{d^2[ce^{\lambda x}]}{dx^2} = c\lambda^2 e^{\lambda x}.$$

Substituting these results into (1) leads to the equation

$$(\lambda^2 + A\lambda + B)e^{\lambda x} = 0.$$

However, the factor $e^{\lambda x}$ is nonvanishing for all x, so after its cancellation this equation is seen to be equivalent to the quadratic equation for λ

$$\lambda^2 + A\lambda + B = 0. \tag{6}$$

When the quadratic equation (6) has two distinct (different) roots λ_1 and λ_2, the functions $y_1(x) = \exp(\lambda_1 x)$ and $y_2(x) = \exp(\lambda_2 x)$ will be linearly independent for all x, because $y_1(x)/y_2(x) = \exp[(\lambda_1 - \lambda_2)x]$ is not constant. Thus, then $\exp(\lambda_1 x)$ and $\exp(\lambda_2 x)$ are linearly independent solutions of (1), so the general solution is

$$y(x) = c_1 \exp(\lambda_1 x) + c_2 \exp(\lambda_2 x), \tag{7}$$

where c_1 and c_2 are arbitrary constants.

It is now necessary to introduce the type of initial conditions that are appropriate for (1). As (1) is a second order differential equation, it relates $y(x)$, $y'(x)$, and $y''(x)$, so it follows that suitable initial conditions will be the specification of $y(x)$ and $y'(x)$ at some point $x = a$. Then the value of $y''(a)$ cannot be assigned arbitrarily, because the differential equation itself will determine its value in terms of $y(a)$ and $y'(a)$. The solution of (1) satisfying these initial conditions can be found from the general solution (7) by determining c_1 and c_2 from the two equations:

initial conditions

Initial condition on y(x)

$$y(a) = c_1 \exp(\lambda_1 a) + c_2 \exp(\lambda_2 a),$$

Initial condition on y′(x)

$$y'(a) = \lambda_1 c_1 \exp(\lambda_1 a) + \lambda_2 c_2 \exp(\lambda_2 a). \tag{8}$$

When we considered systems of linear algebraic equations in Chapter 3, it was shown that equations (8) will determine c_1 and c_2 uniquely if the determinant of the coefficients of c_1 and c_2 is nonvanishing. Thus, the specification of $y(a)$ and $y'(a)$ will be appropriate as initial conditions if

$$\Delta = \begin{vmatrix} \exp(\lambda_1 a) & \exp(\lambda_2 a) \\ \lambda_1 \exp(\lambda_1 a) & \lambda_2 \exp(\lambda_2 a) \end{vmatrix} \neq 0. \tag{9}$$

Expanding the determinant gives $\Delta = (\lambda_2 - \lambda_1)\exp[(\lambda_1 + \lambda_2)a]$. However, by hypothesis $\lambda_1 \neq \lambda_2$, while $\exp[(\lambda_1 + \lambda_2)a]$ never vanishes, so $\Delta \neq 0$. The particular solution satisfying the initial conditions follows by using the values of c_1 and c_2 found from (8) in the general solution (7).

EXAMPLE 6.3 Find the solution of the initial value problem

$$y'' + 4y = 0, \quad \text{if} \quad y(\pi/4) = 1 \quad \text{and} \quad y'(\pi/4) = 1.$$

Solution In Example 6.2 direct substitution has already been used to show that $\cos 2x$ and $\sin 2x$ are linearly independent solutions of the differential equation, so its general solution is

$$y(x) = c_1 \cos 2x + c_2 \sin 2x,$$

from which it follows by differentiation that

$$y'(x) = -2c_1 \sin 2x + 2c_2 \cos 2x.$$

Imposing the initial condition on $y(x)$ at $x = \pi/4$ leads to the following equation that must be satisfied by c_1 and c_2:

$$1 = c_1 \cos \pi/2 + c_2 \sin \pi/2.$$

Similarly, imposing the initial condition on $y'(x)$ at $x = \pi/4$ leads to the second condition that must be satisfied by c_1 and c_2:

$$1 = -2c_1 \sin \pi/2 + 2c_2 \cos \pi/2.$$

$$1 = -2c_1 \sin \pi/2 + 2c_2 \cos \pi/2.$$

These equations have the solution $c_1 = -1/2$ and $c_2 = 1$, so the particular solution satisfying the initial conditions $y(\pi/4) = 1$ and $y'(\pi/4) = 1$ is

$$y(x) = \sin 2x - \frac{1}{2}\cos 2x.$$

The quadratic equation determining the permissible values of λ in the exponential solutions $y_1(x) = \exp(\lambda_1 x)$ and $y_2(x) = \exp(\lambda_2 x)$ of differential equation (1), namely,

$$\lambda^2 + A\lambda + B = 0, \tag{10}$$

characteristic equation

is called the **characteristic equation** of the differential equation. Its two roots,

$$\lambda_1 = \frac{-A + \sqrt{A^2 - 4B}}{2} \quad \text{and} \quad \lambda_2 = \frac{-A - \sqrt{A^2 - 4B}}{2}, \tag{11}$$

are the values of λ to be used in the general solution (7). When the roots λ_1 and λ_2 are real and distinct, the functions

$$y_1(x) = \exp(\lambda_1 x) \quad \text{and} \quad y_2(x) = \exp(\lambda_2 x) \tag{12}$$

are said to form a **basis** for the solution space of (1). This means that the solution of every initial value problem for (1) can be obtained from the linear combination $y(x) = c_1 \exp(\lambda_1 x) + c_2 \exp(\lambda_2 x)$ by assigning suitable values to c_1 and c_2.

A comparison of differential equation (1) and its characteristic equation (10) shows the characteristic equation can be written down immediately from the differential equation by simply replacing y by 1, dy/dx by λ and d^2y/dx^2 by λ^2. It is how a solution depends on the roots

Case (I) (Real and Distinct Roots)

This case corresponds to the condition $A^2 - 4B > 0$, with

$$\lambda_1 = \frac{-A + \sqrt{A^2 - 4B}}{2} \quad \text{and} \quad \lambda_2 = \frac{-A - \sqrt{A^2 - 4B}}{2}. \tag{13}$$

No more need be said about this case because it has already been established that the functions $\exp(\lambda_1 x)$ and $\exp(\lambda_2 x)$ form a basis for the solution space of (1), which thus has the general solution

$$y(x) = c_1 \exp(\lambda_1 x) + c_2 \exp(\lambda_2 x).$$

Case (II) (Complex Conjugate Roots)

This case corresponds to the condition $A^2 - 4B < 0$. A real solution $y(x)$ corresponding to complex conjugate roots λ_1 and λ_2 is only possible if the arbitrary constants c_1 and c_2 are themselves complex conjugates. A routine calculation shows that if $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$, with

$$\alpha = -(1/2)A, \quad \beta = (1/2)(4B - A^2)^{1/2}, \tag{14}$$

Case (II) (Complex Conjugate Roots)

This case corresponds to the condition $A^2 - 4B < 0$. A real solution $y(x)$ corresponding to complex conjugate roots λ_1 and λ_2 is only possible if the arbitrary constants c_1 and c_2 are themselves complex conjugates. A routine calculation shows that if $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$, with

$$\alpha = -(1/2)A, \quad \beta = (1/2)(4B - A^2)^{1/2}, \tag{14}$$

the two corresponding linearly independent solutions are

$$y_1(x) = e^{\alpha x} \cos \beta x \quad \text{and} \quad y_2(x) = e^{\alpha x} \sin \beta x. \tag{15}$$

A basis for the solution space of (1) is formed by the functions $e^{\alpha x} \cos \beta x$ and $e^{\alpha x} \sin \beta x$, corresponding to a general solution of the form

$$y_1(x) = e^{\alpha x}[c_1 \cos \beta x + c_2 \sin \beta x]. \tag{16}$$

The calculation required to establish the form of this result is left as an exercise.

Case (III) (Equal Real Roots)

This case corresponds to the condition $A^2 - 4B = 0$, with

$$\mu = \lambda_1 = \lambda_2 = -(1/2)A. \tag{17}$$

In this case only the one exponential solution

$$y_1(x) = e^{\mu x} \tag{18}$$

can be found.

However, substitution of the function

$$y_2(x) = xe^{\mu x} \tag{19}$$

into the differential equation shows that it is also a solution. The functions $y_1(x)$ and $y_2(x)$ are linearly independent because $y_2(x)/y_1(x) = x$ is not a constant, so in this case a basis for the solution space of (1) is formed by the functions $e^{\mu x}$ and $xe^{\mu x}$, with the corresponding general solution

$$y(x) = (c_1 + c_2 x)e^{\mu x}. \tag{20}$$

summary of types of solution

Summary of the forms of solution of $y'' + Ay' + By = 0$

Characteristic equation: $\lambda^2 + A\lambda + B = 0$

Case (I) $A^2 - 4B > 0$. The general solution is

$$y(x) = c_1 \exp(\lambda_1 x) + c_2 \exp(\lambda_2 x), \quad \text{with}$$

$$\lambda_1 = \frac{-A + \sqrt{A^2 - 4B}}{2} \quad \text{and} \quad \lambda_2 = \frac{-A - \sqrt{A^2 - 4B}}{2}.$$

Case (II) $A^2 - 4B < 0$. The general solution is

$$y_1(x) = e^{\alpha x}[c_1 \cos \beta x + c_2 \sin \beta x], \quad \text{with}$$

$$\alpha = -(1/2)A \quad \text{and} \quad \beta = (1/2)(4B - A^2)^{1/2}.$$

Case (III) $A^2 = 4B$. The general solution is

$$y(x) = (c_1 + c_2 x)e^{\mu x}, \quad \text{with} \quad \mu = -(1/2)A.$$

EXAMPLE 6.4

Find the general solution and hence solve the stated initial value problem for

(i) $y'' + y' - 2y = 0$, with $y(0) = 1$ and $y'(0) = 2$;
(ii) $y'' + 2y' + 4y = 0$, with $y(0) = 2$ and $y'(0) = 1$;
(iii) $y'' + 4y' + 4y = 0$, with $y(0) = 3$ and $y'(0) = 1$.

Solution

(i) The characteristic equation is

$$\lambda^2 + \lambda - 2 = 0,$$

with the roots $\lambda_1 = 1, \lambda_2 = -2$, so this is Case (I). The general solution is

$$y(x) = c_1 e^x + c_2 e^{-2x}.$$

A^2-$4B^2$ = 1-4(-2)= 9 >0 **Case I**

The initial condition $y(0) = 1$ is satisfied if

$$1 = c_1 + c_2,$$

while the initial condition $y'(0) = 2$ is satisfied if

$$2 = c_1 - 2c_2.$$

These equations have the solution $c_1 = 4/3$ and $c_2 = -1/3$, so the solution of the initial value problem is

$$y(x) = (4/3)e^x - (1/3)e^{-2x}.$$

(ii) The characteristic equation is

$$\lambda^2 + 2\lambda + 4 = 0,$$

with $A^2 - 4B = -12$, so this is Case (II) with $\alpha = -1$ and $\beta = \sqrt{3}$. The general solution is

$$y(x) = e^{-x}[c_1\cos(x\sqrt{3}) + c_2\sin(x\sqrt{3})].$$

The initial condition $y(0) = 2$ is satisfied if $2 = c_1$, while the initial condition $y'(0) = 1$ is satisfied if

$$1 = -2 + c_2\sqrt{3}.$$

Solving these equations gives $c_1 = 2$ and $c_2 = \sqrt{3}$, so the solution of the initial value problem is

$$y(x) = e^{-x}[\sqrt{3}\sin(x\sqrt{3}) + 2\cos(x\sqrt{3})].$$

(iii) The characteristic equation is

$$\lambda^2 + 4\lambda + 4 = 0,$$

with $A^2 - 4B = 0$, so this is Case (III) with $\mu = -2$. The general solution is

$$y(x) = (c_1 + c_2 x)e^{-2x}.$$

Using the initial condition $y(0) = 3$ shows that $3 = c_1$, whereas the initial condition $y'(0) = 1$ will be satisfied if

$$1 = -6 + c_2.$$

Solving these equations gives $c_1 = 3$ and $c_2 = 7$, so the solution of the initial value problem is

$$y(x) = (3 + 7x)e^{-2x}.$$ ■

We now formulate the fundamental existence and uniqueness theorem for the homogeneous linear second order constant coefficient differential equation (1). This is a special case of a more general theorem that will be quoted later.

THEOREM 6.1

existence and uniqueness of solutions

Existence and uniqueness of solutions of homogeneous second order constant coefficient equations Let differential equation (1) have two linearly independent solutions $y_1(x)$ and $y_2(x)$. Then, for any $x = x_0$ and numbers μ_1 and μ_2, a unique solution of (1) exists satisfying the initial conditions

$$y(x_0) = \mu_0, \quad y^{(1)}(x_0) = \mu_1.$$

Proof The existence of the solutions $y_1(x)$ and $y_2(x)$ was established when the cases (I), (II), and (III) were examined. The nonvanishing of the determinant Δ in (9) showed c_1 and c_2 to be uniquely determined by the given initial conditions when the roots are real and distinct, so the solution of the initial value problem is also unique. An examination of the form of the determinant Δ in cases (II) and (III) establishes the uniqueness of the solution in the remaining two cases, though the details are left as an exercise. ■

two-point boundary conditions

A different type of problem that can arise with second order equations occurs when the solution is required to satisfy a condition at two distinct points $x = a$ and $x = b$, instead of satisfying two initial conditions. Problems of this type are called **two-point boundary value problems**, because the points a and b can be regarded as boundaries between which the solution is required, and at which it must satisfy given **boundary conditions**. Problems of this type occur in the study of the bending of beams that are supported in different ways at each end, and elsewhere (see Section 8.10).

Typical two-point boundary value problems involve either the specification of $y(x)$ at $x = a$ and at $x = b$, or the specification of $y(x)$ at one boundary and $y'(x)$ at the other one. The most general two point boundary value problem involves finding a solution in the interval $a < x < b$ such that

$$y'' + Ay' + By = 0,$$

subject to the boundary condition at $x = a$

$$\alpha y(a) + \beta y'(a) = \mu,$$

and the boundary condition at $x = b$

$$\gamma y(b) + \delta y'(b) = K,$$

where $\alpha, \beta, \gamma, \delta, \mu$, and K are known constants.

EXAMPLE 6.5

Solve the two-point boundary value problem

$$y'' + 2y' + 17y = 0, \quad \text{with } y(0) = 1 \text{ and } y'(\pi/4) = 0.$$

Solution The characteristic equation is

$$\lambda^2 + 2\lambda + 17 = 0$$

with the complex roots $\lambda_1 = -1 + 4i$ and $\lambda_2 = -1 - 4i$, so the general solution is

$$y(x) = e^{-x}[c_1 \cos 4x + c_2 \sin 4x].$$

At the boundary $x = 0$ the general solution reduces to $1 = c_1$, whereas at the boundary $x = \pi/4$ it reduces to $0 = -e^{-\pi/4} + 4c_2 e^{-\pi/4}$, showing that $c_2 = 1/4$. So the solution of the two-point boundary value problem is

$$y(x) = e^{-x}\left[\cos 4x + \frac{1}{4}\sin 4x\right], \quad \text{for } 0 < x < \pi/4.$$ ■

Summary

This section introduced the homogeneous linear second order constant coefficient equation and explained the importance of the linear independence of solutions. It showed how for this second order equation the general solution can be expressed as a linear combination of the two linearly independent solutions that can always be found. The form of the two linearly independent solutions was shown to depend on the relationship between the roots of the characteristic equation. A fundamental existence and uniqueness theorem was given and the nature of a simple two-point boundary value problem was explained.

EXERCISES 6.1

In Exercises 1 through 4 test the given pairs of functions for linear independence or dependence over the stated intervals.

1. **(a)** $\sinh^2 x, \cosh^2 x$, for all x.
(b) $x + \ln|x|, x + 2\ln|x|$, for $|x| > 0$.
(c) $1 + x, x + x^2$, for all x.

2. **(a)** $\sin x, \cos x$, for all x.
(b) $\sin x \cos x, \sin 2x$, for all x.
(c) e^{2x}, xe^{2x}, for all x.

3. **(a)** $|x|x^2, x^3$, for $-1 < x < 1$.
(b) $\sin x, \tan x$, for $-\pi/4 \le x \le \pi/4$.
(c) $x|x|, x^2$, for $x \ge 0$.

4. **(a)** $\sin x, |\sin x|$, for $\pi \le x \le 2\pi$.
(b) $x^3 - 2x + 4, -4x^3 + 8x - 16$, for all x.
(c) $x + 2|x|, x - 2|x|$ for all x.

Find the general solution of the differential equations in Exercises 5 through 20.

5. $y'' + 3y' - 4y = 0$.
6. $y'' + 2y' + y = 0$.
7. $y'' - 2y' + 2y = 0$.
8. $y'' + 2y' + 2y = 0$.
9. $y'' + 2y' - 3y = 0$.
10. $y'' + 5y' + 4y = 0$.
11. $y'' + 6y' + 9y = 0$.
12. $y'' - 2y' + 4y = 0$.
13. $y'' - 4y' + 5y = 0$.
14. $y'' + 3y' + 3y = 0$.
15. $y'' + 6y' + 25y = 0$.
16. $y'' - 4y' + 20y = 0$.
17. $y'' + 5y' + 4y = 0$.
18. $y'' + 4y' + 5y = 0$.
19. $y'' - 3y' + 3y = 0$.
20. $y'' + y' + y = 0$.

Solve initial value problems in Exercises 21 through 28 using the method of this section, and confirm the solutions for even numbered problems by using computer algebra.

21. $y'' + 5y' + 6y = 0$, with $y(0) = 1$, $y'(0) = 2$.
22. $y'' + 4y' + 5y = 0$, with $y(0) = 1$, $y'(0) = 3$.
23. $y'' + 2y' + 2y = 0$, with $y(0) = 3$, $y'(0) = 1$.
24. $y'' + 6y' + 8y = 0$, with $y(0) = 1$, $y'(0) = 0$.
25. $y'' - 5y' + 6y = 0$, with $y(0) = 2$, $y'(0) = 1$.
26. $y'' - 3y' + 3y = 0$, with $y(0) = 0$, $y'(0) = 2$.
27. $y'' - 3y' - 4y = 0$, with $y(0) = -1$, $y'(0) = 2$.
28. $y'' - 2y' + 3y = 0$, with $y(0) = 1$, $y'(0) = 0$.

Solve the boundary value problems in Exercises 29 through 36 using the method of this section, and confirm the solutions for even-numbered problems by using computer algebra.

29. $y'' + 4y' + 3y = 0$, with $y(0) = 1$, $y'(1) = 0$.
30. $y'' + 4y' + 4y = 0$, with $y(0) = 2$, $y'(1) = 0$.
31. $y'' + 6y' + 9y = 0$, with $y(-1) = 1$, $y'(1) = 0$.
32. $y'' + 4y' + 5y = 0$, with $y(-\pi/2) = 1$, $y'(\pi/2) = 0$.
33. $y'' + 2y' + 26y = 0$, with $y(0) = 1$, $y'(\pi/4) = 0$.
34. $y'' + 2y' + 26y = 0$, with $y(0) = 0$, $y'(\pi/4) = 2$.
35. $y'' + 5y' + 6y = 0$, with $y(0) = 0$, $y'(1) = 1$.
36. $y'' + 2y' - 3y = 0$, with $y(0) = 1$, $y'(1) = 1$.

Theorem 6.1 ensures the existence and uniqueness of solutions of initial value problems for the differential equation in (1), but does not apply to two-point boundary value problems that may have no solution, a unique solution or infinitely many solutions. In Exercises 37 and 38 use the general solution of

$$y'' + y = 0$$

to find if a solution exists and is unique, exists but is nonunique, or does not exist for each set of boundary conditions.

37. **(a)** $y(0) = 0$, $y(\pi) = 0$.
(b) $y(0) = 1$, $y(2\pi) = 2$.
(c) $y'(0) = 1$, $y(\pi/4) = \sqrt{2}$.

38. **(a)** $y(0) = 1$, $y(\pi/2) = 1$.
(b) $y(0) = 0$, $y'(\pi) = 0$.
(c) $y'(0) = 0$, $y'(\pi) = 0$.

39. For what values of λ will the following two-point boundary value problem have infinitely many solutions, and what is the form of these solutions:

$$y'' + \lambda^2 y = 0, \quad \text{with } y(0) = 0,\ y(\pi) = 0.$$

40. A particle moves in a straight line in such a way that its distance x from the origin at time t obeys the differential equation $x'' + x' + x = 0$. Assuming it starts from the origin with speed 30 ft/sec, what will be its distance from the origin, its speed, and its acceleration after $\pi/\sqrt{3}$ seconds?

41. The angular displacement θ of a damped simple pendulum obeys the equation $\theta'' + 2\mu\theta' + (\mu^2 + p^2)\theta = 0$,

Example 3.1.1: General Solution

Solve

$$y'' + 3y' - 4y = 0$$

Solution

The strategy is to search for a solution of the form

$$y = e^{rt}.$$

The reason for this is that long ago some geniuses figured this stuff out and it works.

Now calculate derivatives

$$y' = re^{rt}$$

$$y'' = r^2 e^{rt}.$$

Substituting into the differential equation gives

$$r^2 e^{rt} + 3(re^{rt}) - 4(e^{rt}) = (r^2 + 3r - 4)e^{rt} = 0.$$

Now divide by e^{rt} to get

$$r^2 + 3r - 4 = 0$$

$$(r-1)(r+4) = 0$$

The solutions (roots) to this polynomial are

$$r = 1$$

and

$$r = -4$$

We can conclude that two solutions are

$$y_1 = e^t$$

and

$$y_2 = e^{-4t}$$

Now let

$$L(y) = y'' + 3y' - 4$$

It is easy to verify that if y_1 and y_2 are solutions to

$$L(y) = 0$$

then

$$c_1 y_1 + c_2 y_2$$

is also a solution. More specifically we can conclude that

$$y = c_1 e^t + c_2 e^{-4t}$$

Represents a two dimensional family (vector space) of solutions. Later we will prove that this is the most general description of the solution space.

Solve

$$y'' - y' - 6y = 0$$

with $y(0) = 1$ and $y'(0) = 2$.

Solution

As before we seek solutions of the form

$$y = e^{rt}$$

Now calculate derivatives

$$y' = re^{rt} y'' = r^2 e^{rt}$$

Substituting into the differential equation gives

$$r^2 e^{rt} + (re^{rt}) - 6(e^{rt})$$

$$(r^2 - r - 6)e^{rt} = 0$$

Now divide by e^{rt} to get

$$r^2 - r - 6 = 0$$

$$(r-3)(r+2) = 0$$

We can conclude that two solutions are

$$y_1 = e^{3t}$$

and

$$y_2 = e^{-2t}$$

We can conclude that

$$y = c_1 e^{3t} + C_2 e^{-2t}$$

Represents a two dimensional family (vector space) of solutions. Now use the initial conditions to find that

$$1 = c_1 + c_2$$

We have that

$$y' = 3C_1 e^{3t} - 2C_2 e^{-2t}$$

Plugging in the initial condition with y', gives

$$2 = 3c_1 - 2c_2$$

This is a system of two equations and two unknowns. We can use a matrix to arrive at $c_1 = \frac{4}{5}$ and $C_2 = \frac{1}{5}$

The final solution is

$$y = \frac{4}{5}e^{3t} + \frac{1}{5}e^{-2t}$$

In general for

$$ay'' + by' + cy = 0 \qquad (3.1.5)$$

we call

$$ar^2 + br + c = 0 \qquad (3.1.6)$$

EXAMPLE 1 Solve the equation $y'' + y' - 6y = 0$.

SOLUTION The auxiliary equation is

$$r^2 + r - 6 = (r - 2)(r + 3) = 0$$

whose roots are $r = 2, -3$. Therefore, by (8) the general solution of the given differential equation is

$$y = c_1 e^{2x} + c_2 e^{-3x}$$

We could verify that this is indeed a solution by differentiating and substituting into the differential equation.

EXAMPLE 2 Solve $3\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - y = 0$.

SOLUTION To solve the auxiliary equation $3r^2 + r - 1 = 0$ we use the quadratic formula:

$$r = \frac{-1 \pm \sqrt{13}}{6}$$

Since the roots are real and distinct, the general solution is

$$y = c_1 e^{(-1+\sqrt{13})x/6} + c_2 e^{(-1-\sqrt{13})x/6}$$

CASE II ▫ $b^2 - 4ac = 0$

In this case $r_1 = r_2$; that is, the roots of the auxiliary equation are real and equal. Let's denote by r the common value of r_1 and r_2. Then, from Equations 7, we have

9 $$r = -\frac{b}{2a} \qquad \text{so} \qquad 2ar + b = 0$$

We know that $y_1 = e^{rx}$ is one solution of Equation 5. We now verify that $y_2 = xe^{rx}$ is also a solution:

$$\begin{aligned} ay_2'' + by_2' + cy_2 &= a(2re^{rx} + r^2xe^{rx}) + b(e^{rx} + rxe^{rx}) + cxe^{rx} \\ &= (2ar + b)e^{rx} + (ar^2 + br + c)xe^{rx} \\ &= 0(e^{rx}) + 0(xe^{rx}) = 0 \end{aligned}$$

The first term is 0 by Equations 9; the second term is 0 because r is a root of the auxiliary equation. Since $y_1 = e^{rx}$ and $y_2 = xe^{rx}$ are linearly independent solutions, Theorem 4 provides us with the general solution.

10 If the auxiliary equation $ar^2 + br + c = 0$ has only one real root r, then the general solution of $ay'' + by' + cy = 0$ is

$$y = c_1 e^{rx} + c_2 x e^{rx}$$

EXAMPLE 3 Solve the equation $4y'' + 12y' + 9y = 0$.

SOLUTION The auxiliary equation $4r^2 + 12r + 9 = 0$ can be factored as

$$(2r + 3)^2 = 0$$

so the only root is $r = -\frac{3}{2}$. By (10) the general solution is

$$y = c_1 e^{-3x/2} + c_2 x e^{-3x/2}$$

CASE III ▫ $b^2 - 4ac < 0$

In this case the roots r_1 and r_2 of the auxiliary equation are complex numbers. (See Appendix I for information about complex numbers.) We can write

$$r_1 = \alpha + i\beta \qquad r_2 = \alpha - i\beta$$

where α and β are real numbers. [In fact, $\alpha = -b/(2a)$, $\beta = \sqrt{4ac - b^2}/(2a)$.] Then, using Euler's equation

$$e^{i\theta} = \cos\theta + i\sin\theta$$

from Appendix I, we write the solution of the differential equation as

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x} = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$$

$$\begin{aligned} y &= C_1 e^{r_1 x} + C_2 e^{r_2 x} = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x} \\ &= C_1 e^{\alpha x}(\cos\beta x + i\sin\beta x) + C_2 e^{\alpha x}(\cos\beta x - i\sin\beta x) \\ &= e^{\alpha x}[(C_1 + C_2)\cos\beta x + i(C_1 - C_2)\sin\beta x] \\ &= e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x) \end{aligned}$$

where $c_1 = C_1 + C_2$, $c_2 = i(C_1 - C_2)$. This gives all solutions (real or complex) of the differential equation. The solutions are real when the constants c_1 and c_2 are real. We summarize the discussion as follows.

11 If the roots of the auxiliary equation $ar^2 + br + c = 0$ are the complex numbers $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$, then the general solution of $ay'' + by' + cy = 0$ is

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

EXAMPLE 4 Solve the equation $y'' - 6y' + 13y = 0$.

SOLUTION The auxiliary equation is $r^2 - 6r + 13 = 0$. By the quadratic formula, the roots are

$$r = \frac{6 \pm \sqrt{36 - 52}}{2} = \frac{6 \pm \sqrt{-16}}{2} = 3 \pm 2i$$

By (11) the general solution of the differential equation is

$$y = e^{3x}(c_1 \cos 2x + c_2 \sin 2x)$$

$$y'' + y' - 6y = 0 \qquad y(0) = 1 \qquad y'(0) = 0$$

SOLUTION From Example 1 we know that the general solution of the differential equation is

$$y(x) = c_1 e^{2x} + c_2 e^{-3x}$$

Differentiating this solution, we get

$$y'(x) = 2c_1 e^{2x} - 3c_2 e^{-3x}$$

To satisfy the initial conditions we require that

12 $$y(0) = c_1 + c_2 = 1$$

13 $$y'(0) = 2c_1 - 3c_2 = 0$$

From (13) we have $c_2 = \frac{2}{3}c_1$ and so (12) gives

$$c_1 + \tfrac{2}{3}c_1 = 1 \qquad c_1 = \tfrac{3}{5} \qquad c_2 = \tfrac{2}{5}$$

Thus, the required solution of the initial-value problem is

$$y = \tfrac{3}{5}e^{2x} + \tfrac{2}{5}e^{-3x}$$

EXAMPLE 6 Solve the initial-value problem

$$y'' + y = 0 \qquad y(0) = 2 \qquad y'(0) = 3$$

SOLUTION The auxiliary equation is $r^2 + 1 = 0$, or $r^2 = -1$, whose roots are $\pm i$. Thus $\alpha = 0$, $\beta = 1$, and since $e^{0x} = 1$, the general solution is

$$y(x) = c_1 \cos x + c_2 \sin x$$

Since

$$y'(x) = -c_1 \sin x + c_2 \cos x$$

the initial conditions become

$$y(0) = c_1 = 2 \qquad y'(0) = c_2 = 3$$

Therefore, the solution of the initial-value problem is

$$y(x) = 2 \cos x + 3 \sin x$$

EXAMPLE 7 Solve the boundary-value problem

$$y'' + 2y' + y = 0 \qquad y(0) = 1 \qquad y(1) = 3$$

SOLUTION The auxiliary equation is

$$r^2 + 2r + 1 = 0 \qquad \text{or} \qquad (r + 1)^2 = 0$$

whose only root is $r = -1$. Therefore, the general solution is

$$y(x) = c_1 e^{-x} + c_2 x e^{-x}$$

The boundary conditions are satisfied if

$$y(0) = c_1 = 1$$

$$y(1) = c_1 e^{-1} + c_2 e^{-1} = 3$$

The first condition gives $c_1 = 1$, so the second condition becomes

$$e^{-1} + c_2 e^{-1} = 3$$

Solving this equation for c_2 by first multiplying through by e, we get

$$1 + c_2 = 3e \qquad \text{so} \qquad c_2 = 3e - 1$$

Thus, the solution of the boundary-value problem is

$$y = e^{-x} + (3e - 1)xe^{-x}$$

Summary: Solutions of $ay'' + by' + c = 0$

Roots of $ar^2 + br + c = 0$	General solution
r_1, r_2 real and distinct	$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$
$r_1 = r_2 = r$	$y = c_1 e^{rx} + c_2 x e^{rx}$
r_1, r_2 complex: $\alpha \pm i\beta$	$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$