Order of Partial Differential Equation (PDE)

A partial differential equation is an equation containing an unknown function of two or more variables and its partial derivatives with respect to these variables. The order of a partial differential equations is that of the highest-order derivatives. For example

$y' = e^x \sec y$ has order 1, is non-linear, is separable

$y' - e^x y + 3 = 0$ has order 1, is linear, is not separable

$y' - e^x y = 0$ has order 1, is linear, is separable

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = 0 \quad \text{is a first-order PDE.}$$

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} = 0 \quad \text{is a second-order PDE.}$$

$$\frac{\partial^4 u}{\partial x_1^4} + \frac{\partial^2 u}{\partial x_2^2} - u = 0 \quad \text{is a fourth-order PDE.}$$

$$\left(\frac{\partial u}{\partial x_1}\right)^3 + \frac{\partial u}{\partial x_2} + u^4 = 0 \quad \text{is a first-order PDE.}$$

First order with separated variable

To solve a differential equation is to find a way to eliminate the derivatives in the equation so that the relation between the dependent and the independent variables can be exhibited. For a first-order differential equation, this can be achieved by carrying out an integration. The simplest type of differential equations is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x),$$

where $f(x)$ is a given function of x. We know from calculus that

$$y(x) = \int_a^x f(x')\mathrm{d}x'$$

If an equation can be written in the form

$$f(x)\mathrm{d}x + g(y)\mathrm{d}y = 0$$

the solution can be immediately obtained in the form of

$$\int f(x)\mathrm{d}x + \int g(y)\mathrm{d}y = C.$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$$

can be solved by noting that the equation can be written as

$$y\,\mathrm{d}y + x\,\mathrm{d}x = 0.$$

Therefore the solution is given by

$$\int y\,\mathrm{d}y + \int x\,\mathrm{d}x = C$$

or

$$\frac{1}{2}y^2 + \frac{1}{2}x^2 = C.$$

This general solution can be written as

$$y(x) = (C' - x^2)^{1/2}$$

or equivalently as

$$F(x,y) = C'$$

with

$$F(x,y) = x^2 + y^2.$$

Example5.17. Solving a Separable Differential Equation I.

Solve the differential equation $y' = 2t(25 - y)$.

▼ Solution

This is almost identical to the previous example. As before, $y(t) = 25$ is a solution. If $y \neq 25$,

$$\begin{aligned}
\int \frac{1}{25 - y}\,dy &= \int 2t\,dt \\
(-1)\ln|25 - y| &= t^2 + C_0 \\
\ln|25 - y| &= -t^2 - C_0 = -t^2 + C \\
|25 - y| &= e^{-t^2+C} = e^{-t^2}e^C \\
y - 25 &= \pm e^C e^{-t^2} \\
y &= 25 \pm e^C e^{-t^2} = 25 + Ae^{-t^2}.
\end{aligned}$$

As before, all solutions are represented by $y = 25 + Ae^{-t^2}$, allowing A to be zero.

Example5.18. Solving a Seperable Differential Equation II.

Find the solutions to the differential equation

$$\sec(t)\frac{dy}{dt} - e^{y+\sin(t)} = 0.$$

$$\begin{aligned}\sec(t)\frac{dy}{dt} &= e^{y+\sin(t)}\\ \sec(t)\frac{dy}{dt} &= e^{y}e^{\sin(t)}\\ e^{-y}\,dy &= \frac{e^{\sin(t)}}{\sec(t)}\,dt = \cos(t)e^{\sin(t)}\,dt\end{aligned}$$

Now integrate both sides to obtain

$$\begin{aligned}\int e^{-y}\,dy &= \int \cos(t)e^{\sin(t)}\,dt\\ -e^{-y} &= e^{\sin(t)} + C\\ y &= -\ln\left(D - e^{\sin(t)}\right)\end{aligned}$$

Example5.19. Rate of Change Proportional to Size.

Find the solutions to the differential equation $y' = ky$, which models a quantity y that grows or decays proportionally to its size depending on whether k is positive or negative.

▼ Solution

The constant solution is $y(t) = 0$; of course this will not be the solution to any interesting initial value problem. For the non-constant solutions, we proceed much as before:

$$\begin{aligned}\int \frac{1}{y}\,dy &= \int k\,dt\\ \ln|y| &= kt + C\\ |y| &= e^{kt}e^{C}\\ y &= \pm\, e^{C}e^{kt}\\ y &= Ae^{kt}.\end{aligned}$$

Again, if we allow $A = 0$ this includes the constant solution, and we can simply say that $y = Ae^{kt}$ is the general solution. With an initial value we can easily solve for A to get the solution of the initial value problem. In particular, if the initial value is given for time $t = 0$, $y(0) = y_0$, then $A = y_0$ and the solution is $y = y_0e^{kt}$.

Given the logistic equation $y' = ky(M - y)$,

a. Solve the differential equation for y in terms of t. Answer Solution

We separate variables (assuming $y \neq M$,):

$$\int \frac{dy}{y(M-y)} = \int k\,dt$$

$$\int \left(\frac{1}{y} + \frac{1}{M-y}\right) dy = \int kM\,dt$$

$$\ln|y| - \ln|M-y| = kMt + C$$

$$\ln\left|\frac{M-y}{y}\right| = -kMt - C$$

$$\left|\frac{M-y}{y}\right| = e^{-kMt-C}$$

$$\frac{M-y}{y} = Ae^{-kMt},$$

Therefore, we find that

$$y = \frac{M}{1 + Ae^{-kMt}} \quad \text{with} \quad A = \frac{M - y_0}{y_0}$$

5.5 Homogeneous Equations

homogeneous equation of degree *n*

A function $f(x, y)$ is said to be **algebraically homogeneous of degree *n***, or simply **homogeneous of degree *n***, if $f(tx, ty) = t^n f(x, y)$ for some real number n and all $t > 0$, for $(x, y) \neq (0, 0)$.

EXAMPLE 5.8

(a) If $f(x, y) = x^2 + 3xy + 4y^2$, then $f(tx, ty) = t^2(x^2 + 3xy + 4y^2) = t^2 f(x, y)$, so $f(x, y)$ is homogeneous of degree 2.

(b) If $f(x, y) = \ln|y| - \ln|x|$ for $(x, y) \neq (0, 0)$, then $f(x, y) = \ln|y/x|$, so $f(tx, ty) = f(x, y)$, showing that $f(x, y)$ is homogeneous of degree 0.

(c) If

$$f(x, y) = \frac{x^{3/2} + x^{1/2}y + 3y^{3/2}}{2x^{3/2} - xy^{1/2}}, \text{ then } f(tx, ty) = t^0 f(x, y),$$

showing that $f(x, y)$ is homogeneous of degree 0.

(d) If

$$f(x, y) = x^2 + 4y^2 + \sin(x/y), \text{ then } f(tx, ty) = t^2(x^2 + 4y^2) + \sin(x/y),$$

so $f(x, y)$ is *not* homogeneous, because although both the first group of terms and the last term are homogeneous functions of x and y, they are not both homogeneous of the same degree.

(e) If $f(x, y) = \tan(xy + 1)$, then $f(tx, ty) = \tan(t^2xy + 1)$, so $f(x, y)$ is *not* homogeneous. ■

Homogeneous differential equations

The first order ODE in differential form

$$P(x, y)dx + Q(x, y)dy = 0$$

is called **homogeneous** if P and Q are homogeneous functions of the same degree or, equivalently, if when written in the form

$$\frac{dy}{dx} = f(x, y),$$ the function $f(x, y)$ can be written as $$f(x, y) = g(y/x).$$

The substitution $y = ux$ will reduce either form of the homogeneous equation to an equation involving the independent variable x and the new dependent variable u in which the variables are separable. As with most separable equations the solution can be complicated, and it is often the case that y is determined implicitly in terms of x.

Example: Solve the following Equation

$$(y^2 + 2xy)dx - x^2\,dy = 0.$$

Solution Both terms in the differential equation are homogeneous of degree 2, so the equation itself is homogeneous. Differentiating the substitution $y = ux$ gives

$$\frac{dy}{dx} = u + x\frac{du}{dx}, \quad \text{or} \quad dy = udx + xdu.$$

After substituting for y and dy in the differential equation and cancelling x^2, we obtain the variables separable equation

$$u(u+1)dx = xdu, \quad \text{or} \quad \frac{du}{u(u+1)} = \frac{dx}{x}.$$

This has the general solution

$$u = \frac{Cx}{1 - Cx}, \quad \text{but} \quad y = ux \quad \text{and so } y = \frac{Cx^2}{1 - Cx},$$

where C is an arbitrary constant. In this case the general solution is simple and y is determined explicitly in terms of x. ■

$$\frac{dy}{dx} = \frac{y^2}{xy - x^2}.$$

Solution The equation is homogeneous because it can be written

$$\frac{dy}{dx} = \frac{(y/x)^2}{(y/x) - 1}.$$

Making the substitution $y = ux$, and again using the result $dy/dx = u + xdu/dx$, reduces this to the separable equation

$$u + x\frac{du}{dx} = \frac{u^2}{u - 1}, \text{ or } \left(1 - \frac{1}{u}\right)du = \frac{dx}{x}.$$

Integration gives

$$u - \ln|u| = \ln|x| + \ln|C|,$$

where C is an arbitrary integration constant. Finally, substituting $u = y/x$ and simplifying the result we arrive at the following implicit solution for y:

$$y = Ce^{y/x}.$$ ■

EXERCISES 5.5

In Exercises 1 through 14 find by hand calculation the general solution of the given homogeneous or near-homogeneous equations and confirm the result by using computer algebra.

1. $y' = y/(2x + y)$.
2. $y' = (2xy + y^2)/(3x^2)$.
3. $y' = (2x^2 + y^2)/xy$.
4. $y' = (2xy + y^2)/x^2$.
5. $y' = (x - y)/(x + 2y)$.
6. $y' = (x + 4y)/x$.
7. $y' = (2x + y\cos^2(y/x))/(x\cos^2(y/x))$.
8. $y' = 3y^2/(1 + x^2)$.
9. $y' = (x + y\sin^2(y/x))/(x\sin^2(y/x))$.
10. $y' = 3x\exp(x + 2y)/y$.
11. $y' = (y + 2)/(x + y + 2)$.
12. $y' = (y + 1)/(x + 2y + 2)$.
13. $y' = (x + y + 1)/(x - y + 1)$.
14. $y' = (x - y + 1)/(x + y)$.

5.6 Exact Equations

The so-called *exact* equations have a simple structure, and they arise in many important applications as, for example, in the study of thermodynamics. After definition of an exact equation, a test for exactness will be derived and the general solution of such an equation will be found.

definition of an exact equation

Exact equations

The first order ODE

$$M(x, y)dx + N(x, y)dy = 0$$

is said to be **exact** if a function $F(x, y)$ exists such that the total differential

$$d[F(x, y)] = M(x, y)dx + N(x, y)dy.$$

It follows directly that if

$$M(x, y)dx + N(x, y)dy = 0 \qquad (22)$$

is exact, then the total differential

$$d[F(x, y)] = 0,$$

so the general solution of (22) must be

$$F(x, y) = \text{constant}. \qquad (23)$$

EXAMPLE 5.12

The total differential of $F(x, y) = 3x^3 + 2xy^2 + 4y^3 + 2x$ is

$$\begin{aligned} d[F(x, y)] &= (\partial F/\partial x)dx + (\partial F/\partial y)dy \\ &= (9x^2 + 2y^2 + 2)dx + (4xy + 12y^2)dy, \end{aligned}$$

so the exact differential equation

$$(9x^2 + 2y^2 + 2)dx + (4xy + 12y^2)dy = 0$$

has the general solution

$$3x^3 + 2xy^2 + 4y^3 + 2x = \text{constant}. \qquad ■$$

THEOREM 5.1 **Test for exactness** The differential equation

a simple test for exactness

$$M(x, y)dx + N(x, y)dy = 0$$

is exact if and only if $\partial M/\partial y = \partial N/\partial x$. ■

EXAMPLE 5.13 Test for exactness the differential equations

(a) $\{\sin(xy+1) + xy\cos(xy+1)\}dx + x^2\cos(xy+1)dy = 0.$
(b) $(2x + \sin y)dx + (2x\cos y + y)dy = 0.$

Solution In case (a) $M(x, y) = \sin(xy+1) + xy\cos(xy+1)$ and $N(x, y) = x^2\cos(xy+1)$, and $\partial M/\partial y = \partial N/\partial x$, so the equation is exact.

In case (b) $M(x, y) = 2x + \sin y$ and $N(x, y) = 2x\cos y + y$ but $\partial M/\partial y \neq \partial N/\partial x$, so the equation is not exact. ■

Having established a test for exactness, it remains for us to determine how the general solution of an exact equation can be found. The starting point is the fact that if $F(x, y) =$ constant is a solution of the exact equation

$$M(x, y)dx + N(x, y)dy = 0,$$

EXAMPLE 5.14 Show the following equation is exact and find its general solution:

$$\{3x^2 + 2y + 2\cosh(2x+3y)\}dx + \{2x + 2y + 3\cosh(2x+3y)\}dy = 0.$$

Solution In this equation $M(x, y) = 3x^2 + 2y + 2\cosh(2x+3y)$, and $N(x, y) = 2x + 2y + 3\cosh(2x+3y)$, so as $M_y = N_x = 2 + 6\sinh(2x+3y)$ the equation is exact:

$$F(x, y) = \int M(x, y)dx = \int \{3x^2 + 2y + 2\cosh(2x+3y)\}dx$$
$$= x^3 + 2xy + \sinh(2x+3y) + f(y) + C,$$

and

$$F(x, y) = \int N(x, y)dy = \int \{2x + 2y + 3\cosh(2x+3y)\}dy$$
$$= 2xy + y^2 + \sinh(2x+3y) + g(x) + D.$$

For these two expressions to be identical, we must set $f(y) \equiv y^2$, $g(x) \equiv x^3$, and $D = C$, so $F(x, y)$ is seen to be

$$F(x, y) = x^3 + 2xy + y^2 + \sinh(2x+3y) + C,$$

and so the general solution is

$$x^3 + 2xy + y^2 + \sinh(2x+3y) = C,$$

where as C is an arbitrary constant we have chosen to write C rather than $-C$ on the right of the solution. ■

The first equation yields

$$F(x,y) = 2y + \frac{1}{2}x^2y^2 + p(x).$$

The second equation requires

$$\frac{\partial F(x,y)}{\partial x} = xy^2 + \frac{\mathrm{d}}{\mathrm{d}x}p(x) = xy^2.$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}x}p(x) = 0, \qquad p(x) = k.$$

Thus the solution is

$$F(x,y) = 2y + \frac{1}{2}x^2y^2 + k = C.$$

Combining the two constants, we can write the solution as

$$2y + \frac{1}{2}x^2y^2 = C'.$$

For example, the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{xy^2}{2 + x^2y} = 0$$

can be written in the form

$$(2 + x^2y)\mathrm{d}y + xy^2\mathrm{d}x = 0.$$

Since

$$\frac{\partial}{\partial x}(2 + x^2y) = 2xy, \qquad \frac{\partial}{\partial y}(xy^2) = 2xy$$

are equal, the differential equation is exact. Therefore, we can find the general solution in the form of

$$F(x,y) = C$$

with

$$\frac{\partial F(x,y)}{\partial y} = 2 + x^2y, \qquad \frac{\partial F(x,y)}{\partial x} = xy^2.$$

Exercises

Show that the following differential equations are exact and find the general solutions:

(a) $(2xy - \cos x)\mathrm{d}x + (x^2 - 1)\mathrm{d}y = 0$,

(b) $(2x + e^y)\mathrm{d}x + xe^y\mathrm{d}y = 0$.

Ans. (a) $x^2y - \sin x - y = c$, (b) $x^2 + xe^y = c$.

5.7 Linear First Order Equations

The **standard form** of the **linear first order differential equation** is

standard form of linear first order equation

$$\frac{dy}{dx} + P(x)y = Q(x), \tag{25}$$

steps used when solving a linear first order equation

Rule for solving linear first order equations

STEP 1 If the equation is not in standard form and is written

$$a(x)\frac{dy}{dx} + b(x)y = c(x),$$

divide by $a(x)$ to bring it to the standard form

$$\frac{dy}{dx} + P(x)y = Q(x),$$

with $P(x) = b(x)/a(x)$ and $Q(x) = c(x)/a(x)$

STEP 2 Find the integrating factor

$$\mu(x) = \exp\left\{\int P(x)dx\right\}.$$

STEP 3 Rewrite the original differential equation in the form

$$\frac{d(\mu y)}{dx} = \mu Q(x).$$

STEP 4 Integrate the equation in Step 3 to obtain

$$\mu(x)y(x) = \int \mu(x)Q(x)dx + C.$$

STEP 5 Divide the result of Step 4 by $\mu(x)$ to obtain the required general solution of the linear first order differential equation in Step 1.

STEP 6 If an initial condition $y(x_0) = y_0$ is given, the required solution of the i.v.p. is obtained by choosing the arbitrary constant C in the general solution found in Step 5 so that $y = y_0$ when $x = x_0$.

$$y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)\,q(x)\mathrm{d}x + C\right]$$

EXAMPLE 5.15 Solve the initial value problem

$$\cos x \frac{dy}{dx} + y = \sin x, \text{ subject to the initial condition } y(0) = 2.$$

Solution We follow the steps in the above rule.

STEP 1 When written in standard form the equation becomes

$$\frac{dy}{dx} + \frac{1}{\cos x} y = \tan x,$$

so $P(x) = 1/\cos x$ and $Q(x) = \tan x$.

$$\frac{dy}{dx} + P(x)y = Q(x),$$

STEP 2 The integrating factor

$$\mu(x) = \exp\left\{\int \frac{dx}{\cos x}\right\} = \exp\{\ln|\sec x + \tan x|\}$$

$$= \sec x + \tan x = \frac{1 + \sin x}{\cos x}.$$

STEP 3 The original differential equation can now be written

$$\frac{d}{dx}\left[\left(\frac{1+\sin x}{\cos x}\right) y(x)\right] = \left(\frac{1+\sin x}{\cos x}\right)\tan x.$$

STEP 4 Integrating the result of Step 3 gives

$$\left(\frac{1+\sin x}{\cos x}\right) y(x) = \int \left(\frac{1+\sin x}{\cos x}\right)\tan x dx + C$$

$$= \int \sec x \tan x dx + \int \tan^2 x dx + C$$

$$= \sec x + \tan x - x + C = \frac{1+\sin x}{\cos x} - x + C.$$

STEP 5 Dividing the result of Step 4 by the integrating factor $\mu(x) = (1 + \sin x)/\cos x$ shows that the required general solution is

$$y(x) = \frac{C \cos x}{1 + \sin x} + 1 - \frac{x \cos x}{1 + \sin x},$$

Note:

$$\int \sec\theta \, d\theta = \begin{cases} \frac{1}{2}\ln\frac{1+\sin\theta}{1-\sin\theta} + C \\ \ln\left|\sec\theta + \tan\theta\right| + C \\ \ln\left|\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)\right| + C \end{cases}$$

Example 5.2.1. Find the general solution of the following differential equation:

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + (1+x)y = \mathrm{e}^x.$$

Solution 5.2.1. This is a linear differential equation of first-order

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1+x}{x}y = \frac{\mathrm{e}^x}{x}.$$

The integrating factor is given by

$$\mu(x) = \mathrm{e}^{\int \frac{1+x}{x}\mathrm{d}x}.$$

Since

$$\int^x \frac{1+x}{x}\mathrm{d}x = \int \left(\frac{1}{x}+1\right)\mathrm{d}x = \ln x + x,$$

$$\mu(x) = \mathrm{e}^{\ln x + x} = x\mathrm{e}^x.$$

It follows that:

$$y = \frac{1}{x\mathrm{e}^x}\left[\int x\mathrm{e}^x \frac{\mathrm{e}^x}{x}\mathrm{d}x + C\right]$$

$$= \frac{1}{x\mathrm{e}^x}\left[\int \mathrm{e}^{2x}\mathrm{d}x + C\right] = \frac{1}{x\mathrm{e}^x}\left[\frac{\mathrm{e}^{2x}}{2} + C\right].$$

Therefore the solution is given by

$$y = \frac{\mathrm{e}^x}{2x} + C\frac{\mathrm{e}^{-x}}{x}.$$

►*Solve*

$$\frac{dy}{dx} + 2xy = 4x.$$

The integrating factor is given immediately by

$$\mu(x) = \exp\left\{\int 2x\,dx\right\} = \exp x^2.$$

Multiplying through the ODE by $\mu(x) = \exp x^2$ and integrating, we have

$$y\exp x^2 = 4\int x\exp x^2\,dx = 2\exp x^2 + c.$$

The solution to the ODE is therefore given by $y = 2 + c\exp(-x^2)$. ◄

Exercises

Solve the following differential equations by first finding an integrating factor:

(a) $2(y^3-2)\mathrm{d}x + 3xy^2\mathrm{d}y = 0,$

(b) (b) $(y+x^4)\mathrm{d}x - x\mathrm{d}y = 0.$

Ans. (a) $\mu = x, \quad x^2y^3 - 2x^2 = c,$ (b) $\mu = 1/x^2, \quad \frac{x^3}{3} - \frac{y}{x} = c.$

Table 2.1

Group of terms	Integrating factor $I(x, y)$	Exact differential $dy(x, y)$
$y\,dx - x\,dy$	$-\frac{1}{x^2}$	$\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$
$y\,dx - x\,dy$	$\frac{1}{y^2}$	$\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$
$y\,dx - x\,dy$	$-\frac{1}{xy}$	$\frac{x\,dy - y\,dx}{xy} = d\left(\ln\frac{y}{x}\right)$
$y\,dx - x\,dy$	$-\frac{1}{x^2 + y^2}$	$\frac{x\,dy - y\,dx}{x^2 + y^2} = d\left(\arctan\frac{y}{x}\right)$
$y\,dx + x\,dy$	$\frac{1}{xy}$	$\frac{y\,dx + x\,dy}{xy} = d(\ln xy)$
$y\,dx + x\,dy$	$\frac{1}{(xy)^n}, \quad n > 1$	$\frac{y\,dx + x\,dy}{(xy)^n} = d\left[\frac{-1}{(n-1)(xy)^{n-1}}\right]$
$y\,dy + x\,dx$	$\frac{1}{x^2 + y^2}$	$\frac{y\,dy + x\,dx}{x^2 + y^2} = d\left[\frac{1}{2}\ln(x^2 + y^2)\right]$
$y\,dy + x\,dx$	$\frac{1}{(x^2 + y^2)^n}, \quad n > 1$	$\frac{y\,dy + x\,dx}{(x^2 + y^2)^n} = d\left[\frac{-1}{2(n-1)(x^2 + y^2)^{n-1}}\right]$
$ay\,dx + bx\,dy$ (a, b constants)	$x^{a-1}y^{b-1}$	$x^{a-1}y^{b-1}(ay\,dx + bx\,dy) = d(x^a y^b)$

5.8 The Bernoulli Equation

The **Bernoulli equation** is a nonlinear first order differential equation with the standard form

standard form of the Bernoulli equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n, \quad (n \neq 1). \tag{34}$$

The substitution

$$u = y^{1-n} \tag{35}$$

reduces (34) to the linear first order ODE

$$\frac{1}{(1-n)}\frac{du}{dx} + P(x)u = Q(x), \tag{36}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)y^n$$

is known as Bernoulli equations, named after Swiss mathematician James Bernoulli (1654–1705). This is a nonlinear differential equation if $n \neq 0$ *or* 1. However, it can be transformed into a linear equation by multiplying both sides with a factor $(1-n)y^{-n}$

$$(1-n)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x} + (1-n)p(x)y^{1-n} = (1-n)q(x).$$

Since

$$(1-n)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(y^{1-n}),$$

the last equation can be written as

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^{1-n}) + (1-n)p(x)y^{1-n} = (1-n)q(x),$$

which is a first-order linear equation in terms of y^{1-n}. This equation can be solved for y^{1-n}, from which the solution of the original equation can be obtained.

Example 5.2.2. Find the solution of

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x}y = x^2y^3$$

with the condition $y(1) = 1$.

Solution 5.2.2. This a Bernoulli equation of $n = 3$. Multiplying this equation by $(1-3)y^{-3}$, we have

$$-2y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x} - 2\frac{1}{x}y^{-2} = -2x^2,$$

which can be written as

$$\frac{\mathrm{d}}{\mathrm{d}x}y^{-2} - \frac{2}{x}y^{-2} = -2x^2.$$

This equation is first-order in y^{-2} and can be solved by multiplying it with an integrating factor μ,

$$\mu = \mathrm{e}^{\int(-\frac{2}{x})\mathrm{d}x} = \mathrm{e}^{-2\ln x} = \frac{1}{x^2}.$$

Thus

$$\frac{1}{x^2}y^{-2} = \int \frac{1}{x^2}(-2x^2)\mathrm{d}x + C = -2x + C.$$

At $x = 1$, $y = 1$, therefore

$$1 = -2 + C, \qquad C = 3.$$

Hence the specific solution of the original nonlinear linear differential equation is

$$y^{-2} = -2x^3 + 3x^2$$

2nd Method

Suppose that $u = y^{1-n}$

$$\frac{dy}{dx} + \frac{1}{x} y = x^2 y^3 \Rightarrow \div y^3$$

$$y^{-3} \frac{dy}{dx} + \frac{1}{x} y^{-2} = x^2 \qquad (2)$$

$$n = 3, \quad \frac{du}{dy} = y^{-2} \Rightarrow du = -2y^{-3} dy$$

$$\frac{du}{dx} = -2y^{-3} \frac{dy}{dx}$$

$$y^{-3} \frac{dy}{dx} = \frac{-1}{2} \frac{du}{dx}$$

$$From \;\; Eq.(1) \Rightarrow \frac{-1}{2} \frac{du}{dx} + \frac{1}{x} u = x^2 \quad multiply \;\; by \;\; -2$$

$$\frac{du}{dx} - \frac{2}{x} u = -2x^2$$

$$Compare\ With \qquad \frac{du}{dx} + P(x)u = Q(x)$$

$$P(x) = \frac{-2}{x}, \quad Q(x) = -2x^2$$

$$\mu = e^{\int P(x)dx} = e^{-2\int \frac{1}{x} dx} = e^{-2\ln(x)} = e^{\ln(x^{-2})} = x^{-2}$$

$$Multiply \;\; Eq.(2) \;\; by \;\; x^{-2}$$

$$x^{-2} \frac{du}{dx} - 2x^{-3} u = -2$$

$$\frac{d}{dx}\left(x^{-2} u\right) = -2 \Rightarrow \int d(x^{-3} u) = -2\int dx$$

$$x^{-3} u = -2x + c$$

$$x^{-3} y^{-2} = -2x + c$$

Full worked solutions

Exercise 1.

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

DIVIDE by y^n: $\frac{1}{y^n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)$

SET $z = y^{1-n}$: i.e. $\frac{dz}{dx} = (1-n)y^{(1-n-1)}\frac{dy}{dx}$

i.e. $\frac{1}{(1-n)}\frac{dz}{dx} = \frac{1}{y^n}\frac{dy}{dx}$

SUBSTITUTE $\frac{1}{(1-n)}\frac{dz}{dx} + P(x)z = Q(x)$

i.e. $\boxed{\frac{dz}{dx} + P_1(x)z = Q_1(x)}$ linear in z

where $P_1(x) = (1-n)P(x)$
$Q_1(x) = (1-n)Q(x)$.

Solve the following Bernoulli differential equations:

Exercise 2.

$$\frac{dy}{dx} - \frac{1}{x}y = xy^2$$

Exercise 3.

$$\frac{dy}{dx} + \frac{y}{x} = y^2$$

Exercise 4.

$$\frac{dy}{dx} + \frac{1}{3}y = e^x y^4$$

Exercise 5.

$$x\frac{dy}{dx} + y = xy^3$$

Exercise 6.

$$\frac{dy}{dx} + \frac{2}{x}y = -x^2 \cos x \cdot y^2$$

EXERCISE 7.

$$2\frac{dy}{dx} + \tan x \cdot y = \frac{(4x+5)^2}{\cos x} y^3$$

EXERCISE 8.

$$x\frac{dy}{dx} + y = y^2 x^2 \ln x$$

EXERCISE 9.

$$\frac{dy}{dx} = y \cot x + y^3 \text{cosec} x$$

Exercise 2.

This is of the form $\frac{dy}{dx} + P(x)y = Q(x)y^n$ where

$$\text{where} \quad P(x) = -\frac{1}{x}$$
$$Q(x) = x$$
$$\text{and} \quad n = 2$$

DIVIDE by y^n: i.e. $\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{x}y^{-1} = x$

SET $z = y^{1-n} = y^{-1}$: i.e. $\frac{dz}{dx} = -y^{-2}\frac{dy}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$

$$\therefore \quad -\frac{dz}{dx} - \frac{1}{x}z = x$$

$$\text{i.e.} \quad \frac{dz}{dx} + \frac{1}{x}z = -x$$

Integrating factor, $\text{IF} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

$$\therefore \quad x\frac{dz}{dx} + z = -x^2$$

$$\text{i.e.} \quad \frac{d}{dx}[x \cdot z] = -x^2$$

$$\text{i.e.} \quad xz = -\int x^2 dx$$

$$\text{i.e.} \quad xz = -\frac{x^3}{3} + C$$

Use $z = \frac{1}{y}$: $\frac{x}{y} = -\frac{x^3}{3} + C$

Exercise 3.

This is of the form $\dfrac{dy}{dx} + P(x)y = Q(x)y^n$

where $P(x) = \frac{1}{x}$,

$Q(x) = 1$,

and $n = 2$

DIVIDE by y^n: i.e. $\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = 1$

SET $z = y^{1-n} = y^{-1}$: i.e. $\frac{dz}{dx} = -1 \cdot y^{-2}\frac{dy}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$

$\therefore \quad -\frac{dz}{dx} + \frac{1}{x}z = 1$

i.e. $\frac{dz}{dx} - \frac{1}{x}z = -1$

Integrating factor, $\text{IF} = e^{-\int \frac{dx}{x}} = e^{-\ln x} = e^{\ln x^{-1}} = \dfrac{1}{x}$

$\therefore \quad \frac{1}{x}\frac{dz}{dx} - \frac{1}{x^2}z = -\frac{1}{x}$

i.e. $\frac{d}{dx}\left[\frac{1}{x} \cdot z\right] = -\frac{1}{x}$

i.e. $\frac{1}{x} \cdot z = -\int \frac{dx}{x}$

i.e. $\frac{z}{x} = -\ln x + C$

Use $z = \frac{1}{y}$: $\frac{1}{yx} = C - \ln x$

i.e. $\frac{1}{y} = x(C - \ln x)$.

Return to Exercise 3

Exercise 4.

This of the form $\frac{dy}{dx} + P(x)y = Q(x)y^n$

$$\begin{array}{llll} \text{where} & P(x) & = & \frac{1}{3} \\ & Q(x) & = & e^x \\ \text{and} & n & = & 4 \end{array}$$

DIVIDE by y^n: i.e. $\frac{1}{y^4}\frac{dy}{dx} + \frac{1}{3}y^{-3} = e^x$

SET $z = y^{1-n} = y^{-3}$: i.e. $\frac{dz}{dx} = -3y^{-4}\frac{dy}{dx} = -\frac{3}{y^4}\frac{dy}{dx}$

$$\therefore \quad -\frac{1}{3}\frac{dz}{dx} + \frac{1}{3}z = e^x$$

$$\text{i.e.} \quad \frac{dz}{dx} - z = -3e^x$$

Integrating factor, $\text{IF} = e^{-\int dx} = e^{-x}$

$$\therefore \quad e^{-x}\frac{dz}{dx} - e^{-x}z = -3e^{-x}\cdot e^x$$

$$\text{i.e.} \quad \frac{d}{dx}[e^{-x}\cdot z] = -3$$

$$\text{i.e.} \quad e^{-x}\cdot z = \int -3\,dx$$

$$\text{i.e.} \quad e^{-x}\cdot z = -3x + C$$

Use $z = \frac{1}{y^3}$: $e^{-x}\cdot\frac{1}{y^3} = -3x + C$

$$\text{i.e.} \quad \frac{1}{y^3} = e^x(C - 3x)\,.$$

Return to Exercise 4

Exercise 5.

Bernoulli equation: $\dfrac{dy}{dx} + \dfrac{y}{x} = y^3$ with $P(x) = \dfrac{1}{x}, Q(x) = 1, n = 3$

DIVIDE by y^n i.e. y^3: $\quad \frac{1}{y^3}\frac{dy}{dx} + \frac{1}{x}y^{-2} = 1$

SET $z = y^{1-n}$ i.e. $z = y^{-2}$: $\quad \frac{dz}{dx} = -2y^{-3}\frac{dy}{dx}$

i.e. $-\frac{1}{2}\frac{dz}{dx} = \frac{1}{y^3}\frac{dy}{dx}$

$\therefore \quad -\frac{1}{2}\frac{dz}{dx} + \frac{1}{x}z = 1$

i.e. $\frac{dz}{dx} - \frac{2}{x}z = -2$

Integrating factor, $\quad \text{IF} = e^{-2\int \frac{dx}{x}} = e^{-2\ln x} = e^{\ln x^{-2}} = \dfrac{1}{x^2}$

$\therefore \quad \frac{1}{x^2}\frac{dz}{dx} - \frac{2}{x^3}z = -\frac{2}{x^2}$

i.e. $\frac{d}{dx}\left[\frac{1}{x^2}z\right] = -\frac{2}{x^2}$

i.e. $\frac{1}{x^2}z = (-2)\cdot(-1)\frac{1}{x} + C$

i.e. $z = 2x + Cx^2$

Use $z = \frac{1}{y^2}$: $\quad y^2 = \frac{1}{2x+Cx^2}$.

Return to Exercise 5

5. Standard integrals

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1} \quad (n \neq -1)$	$[g(x)]^n g'(x)$	$\frac{[g(x)]^{n+1}}{n+1} \quad (n \neq -1)$
$\frac{1}{x}$	$\ln\lvert x\rvert$	$\frac{g'(x)}{g(x)}$	$\ln\lvert g(x)\rvert$
e^x	e^x	a^x	$\frac{a^x}{\ln a} \quad (a > 0)$
$\sin x$	$-\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$-\ln\lvert\cos x\rvert$	$\tanh x$	$\ln\cosh x$
$\operatorname{cosec} x$	$\ln\left\lvert\tan\frac{x}{2}\right\rvert$	$\operatorname{cosech} x$	$\ln\left\lvert\tanh\frac{x}{2}\right\rvert$
$\sec x$	$\ln\lvert\sec x + \tan x\rvert$	$\operatorname{sech} x$	$2\tan^{-1} e^x$
$\sec^2 x$	$\tan x$	$\operatorname{sech}^2 x$	$\tanh x$
$\cot x$	$\ln\lvert\sin x\rvert$	$\coth x$	$\ln\lvert\sinh x\rvert$
$\sin^2 x$	$\frac{x}{2} - \frac{\sin 2x}{4}$	$\sinh^2 x$	$\frac{\sinh 2x}{4} - \frac{x}{2}$
$\cos^2 x$	$\frac{x}{2} + \frac{\sin 2x}{4}$	$\cosh^2 x$	$\frac{\sinh 2x}{4} + \frac{x}{2}$

$f(x)$	$\int f(x)\,dx$	$f(x)$	$\int f(x)\,dx$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$ $(a > 0)$	$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert \quad (0<\lvert x\rvert<a)$
		$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert \quad (\lvert x\rvert > a>0)$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}$ $(-a < x < a)$	$\frac{1}{\sqrt{a^2+x^2}}$	$\ln\left\lvert\frac{x+\sqrt{a^2+x^2}}{a}\right\rvert \quad (a > 0)$
		$\frac{1}{\sqrt{x^2-a^2}}$	$\ln\left\lvert\frac{x+\sqrt{x^2-a^2}}{a}\right\rvert \quad (x>a>0)$
$\sqrt{a^2-x^2}$	$\frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{a^2-x^2}}{a^2}\right]$	$\sqrt{a^2+x^2}$	$\frac{a^2}{2}\left[\sinh^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{a^2+x^2}}{a^2}\right]$
		$\sqrt{x^2-a^2}$	$\frac{a^2}{2}\left[-\cosh^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{x^2-a^2}}{a^2}\right]$